The tracking error rate of the Delta-Gamma hedging strategy

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Delta Hedging Strategy (DHS)

- Risky assets $S$ (with Black-Scholes model)
- Option to be hedged: $u(t, S) := \mathbb{E}_Q(e^{-r(T-t)}g(S_T)|S_t = S)$
- Rebalancing dates: $\pi := \{0 = t_0 < \cdots < t_i < \cdots < t_N = T\}$
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**Delta Hedging Strategy (DHS)** $\equiv$ hold $\delta_{t_i}$ risky assets between $t_i$ et $t_{i+1}$ such that:

Portfolio value at time $t$: $V^{\Delta,N}(t, S_t) = \delta^0_t S^0_t + \delta_t S_t$

$$\partial_S V^{\Delta,N} = \partial_S u \implies \delta_t = \partial_S u(t, S_t)$$
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The (discounted) Delta tracking error

$$\bar{\Delta}^{\Delta,N} := e^{-rT}(V_T^{\Delta,N} - g(S_T))$$

$$\bar{\Delta}^{\Delta,N}_N = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (\delta_{t_i} - \delta_t) d\bar{S}_t.$$
Effect of the payoff function smoothness

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  - For $g(x) = (x - K)_+$ (or any Lipschitz continuous $g$),
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    \]
  - For \( g(x) = 1_{x \geq K} \),
    \[
    \left( \mathbb{E} |\bar{\mathcal{E}}_N^A|^2 \right)^{\frac{1}{2}} \sim N^{-\frac{1}{4}}.
    \]
In Geiss (2002) and Gobet and Makhlouf (2008): generalization:

\[ \text{For } g \in L^2, \alpha, \alpha' \in (0, 1), (E|\bar{E} \Delta N|)^{1/2} = O(N^{-\alpha^2/2}). \]

Moreover, one can reach the order \( N^{-1/2} \) thanks to a convenient choice of a non-regular time net.
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For \( g \in L_{2,\alpha}, \alpha \in (0, 1], \)

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(\mathbb{E}|\bar{\mathcal{E}}_N|^{2})^{\frac{1}{2}} = \mathcal{O}(N^{-\frac{\alpha}{2}}).
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In Geiss (2002) and Gobet and Makhlof (2008) : generalization :

- For $g \in L_{2,\alpha}$, $\alpha \in (0, 1]$, 

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Moreover, one can reach the order $N^{-\frac{1}{2}}$ thanks to a convenient choice of a non regular time net.

Both the payoff function regularity and the time net choice have an effect on the convergence order of the Delta hedging error.
The Delta-Gamma Hedging Strategy (DGHS) hold, between $t_i$ and $t_{i+1}$, risky assets $S$ and $\delta_t^C$ of another instrument whose price is $(C(t, S_t))_{0 \leq t \leq T}$: (in dim 1)

$$V^{\Delta \Gamma, N}(t, S_t) = \delta^0_t S^0 + \delta_t S_t + \delta^C_t C(t, S_t)$$
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The Delta-Gamma Hedging Strategy (DGHS) hold, between \( t_i \) and \( t_{i+1} \), \( \delta_{t_i} \) risky assets \( S \) and \( \delta^C_{t_i} \) of another instrument whose price is \( (C(t, S_t))_{0 \leq t \leq T} \) : (in dim 1)

\[
V^{\Delta \Gamma, N}(t, S_t) = \delta^0_{t} S^0_{t} + \delta_{t} S_{t} + \delta^C_{t} C(t, S_t)
\]

\[ \partial_S V^{\Delta \Gamma, N} = \partial_S u \quad \text{and} \quad \partial^2_S V^{\Delta \Gamma, N} = \partial^2_S u \text{ yield (in dim 1)} \]

\[
\delta^C_{t_i} := \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})}, \quad \delta_{t_i} := \partial_S u(t_i, S_{t_i}) - \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})} \partial_S C(t_i, S_{t_i}).
\]
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$$V^{\Delta \Gamma, N}(t, S_t) = \delta^0_t S^0_t + \delta_t S_t + \delta^C_tC(t, S_t)$$

$\partial_S V^{\Delta \Gamma, N} = \partial_S u$ and $\partial^2_S V^{\Delta \Gamma, N} = \partial^2_S u$ yield (in dim 1)

$$\delta^C_{t_i} := \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})}, \quad \delta_{t_i} := \partial_S u(t_i, S_{t_i}) - \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})} \partial_S C(t_i, S_{t_i}).$$

**Our goal**: study, in dimension $d$,

- the link between the order of $\bar{\delta}^{\Delta \Gamma}_N$ and the payoff regularity
- the effect of the rebalancing dates choice
Assets

\[
\begin{cases}
S_0^j = s_0^j, \\
\text{d}S_t^j = \mu_j S_t^j \text{d}t + \sigma_j S_t^j \text{d}\hat{W}_t^j,
\end{cases}
\]

\(\hat{W} = (\hat{W}^1, ..., \hat{W}^d)\) is a Brownian motion under the historical probability \(\mathbb{P}\).

\(\langle \hat{W}^j, \hat{W}^k \rangle_t = \rho_{j,k} t\), and the matrix \((\rho_{j,k})_{1 \leq j,k \leq d}\) has full rank.

Risk-neutral probability \(\mathbb{Q}\):

- \(\lambda_j = \frac{\mu_j - r}{\sigma_j}\)
- \((\hat{W}_t^j := \hat{W}_t^j + \lambda_j t)_{1 \leq j \leq d}\) is a \(\mathbb{Q}\)-Brownian motion
Hedging instruments: for $0 \leq j < k \leq d$,

$$C_{j,k}(t, S_j, S_k) := \mathbb{E}_Q \left[ e^{-r(T_2-t)}(S_{T_2}^k - K_{j,k}S_{T_2}^j) + |S_t^j = S_j, S_t^k = S_k \right],$$

($\rightarrow$ closed BS and Margrabe formulas).
Hedging instruments: for $0 \leq j < k \leq d$, 
\[ C_{j,k}(t, S_j, S_k) := \mathbb{E}_Q \left[ e^{-r(T_2-t)}(S_{k,T_2}^k - K_{j,k} S_{j,T_2}^j) + |S_t^j = S_j, S_t^k = S_k \right], \]
(\rightarrow closed BS and Margrabe formulas).

Hedging ratios:
- $\delta_{j,k}^{t,i}$ (1 $\leq j < k \leq d$, Exchange options)
- $\delta_{0,l}^{0,i}$ (1 $\leq l \leq d$, Call options)
- $\delta_{1,l}^{0,i}$ (1 $\leq l \leq d$, assets).

\[ \text{▶ with almost similar definitions to those in dim 1.} \]
The option to hedge:

\[ u(t,S) := \mathbb{E}_Q \left[ e^{-r(T-t)} g(S_T) | S_t = S \right], \text{ with} \]
\[ S = (S_1, \ldots, S_d) \in \mathbb{R}_+^d. \]
The option to hedge:

\[ u(t, S) := \mathbb{E}_Q \left[ e^{-r(T-t)} g(S_T) \mid S_t = S \right], \text{ with} \]
\[ S = (S^1, \ldots, S^d) \in \mathbb{R}^d_+. \]

Payoff: \( \mathbb{E}_P |g(S_T)|^{2p_0} < \infty, \) for some \( p_0 > 1. \)
For $l, m, n = 1...d$, we define

$$\bar{u}(t) := e^{-rt} u(t, S_t) = \mathbb{E}_Q \left[ e^{-rT} g(S_T) | \mathcal{F}_t \right];$$

$$\bar{u}_l^{(1)}(t) := e^{-rt} \sigma_l S_t \partial_l u(t);$$

$$\bar{u}_l^{(2)}(t) := e^{-rt} \sigma_l \sigma_m S_t S_t \partial_{l, m}^2 u(t);$$

$$\bar{u}_l^{(3)}(t) := e^{-rt} \sigma_l \sigma_m \sigma_n S_t S_t S_t \partial_{l, m, n}^3 u(t).$$

And similar definitions with $\bar{C}^{j, k}(t)$ (for $0 \leq j < k \leq d$ et $l, m, n = 1...d$).
For $l, m, n = 1\ldots d$, we define

\[
\bar{u}(t) := e^{-rt} u(t, S_t) = \mathbb{E}_\mathbb{Q} \left[ e^{-rT} g(S_T) | \mathcal{F}_t \right];
\]

\[
\bar{u}^{(1)}_l(t) := e^{-rt} \sigma_l S^l_t \partial_l u(t);
\]

\[
\bar{u}^{(2)}_{l,m}(t) := e^{-rt} \sigma_l \sigma_m S^l_t S^m_t \partial^2_{l,m} u(t);
\]

\[
\bar{u}^{(3)}_{l,m,n}(t) := e^{-rt} \sigma_l \sigma_m \sigma_n S^l_t S^m_t S^n_t \partial^3_{l,m,n} u(t).
\]

And similar definitions with $\bar{C}^{j,k}(t)$ (for $0 \leq j < k \leq d$ et $l, m, n = 1\ldots d$).

▶ **Q-Martingales.**

▶ enable tricky calculus of the Itô decompositions.
Theorem

\[ E_N^{\Delta \Gamma} (g, \pi) = 
- \sum_{i=0}^{N-1} \sum_{l,m,n=1}^{d} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left( \bar{u}_{l,m,n}^{(3)}(r) + R_{l,m,n}^{i,(3)}(r) \right) \, dW_r \, dW_s \, dW_t. \]
Tracking error decomposition

**Theorem**

\[
\bar{\mathcal{E}}_N^{\Delta \Gamma} (g, \pi) = \\
- \sum_{i=0}^{N-1} \sum_{l,m,n=1}^d \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \int_{t_i}^{s} \left( \bar{u}_{l,m,n}^{(3)} (r) + R_{l,m,n}^{i,(3)} (r) \right) \, dW_{r} \, dW_{m} \, dW_{l}.
\]

**NB.** For DHS : \( \bar{\mathcal{E}}_N^{\Delta} (g, \pi) = \\
- \sum_{i=0}^{N-1} \sum_{l,m=1}^d \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \left( \bar{u}_{l,m}^{(2)} (s) + R_{l,m}^{i,(2)} (s) \right) \, dW_{s} \, dW_{t}.
\]
Tracking error decomposition

**Theorem**

\[
\mathcal{E}_N^{\Delta \Gamma} (g, \pi) = - \sum_{i=0}^{N-1} \sum_{l,m,n=1}^{d} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \int_{t_i}^{s} \left( \bar{u}^{(3)}_{l,m,n}(r) + R^{i,(3)}_{l,m,n}(r) \right) dW_r dW_m dW_t.
\]

**NB.** For DHS : \( \mathcal{E}_N^{\Delta}(g, \pi) = - \sum_{i=0}^{N-1} \sum_{l,m=1}^{d} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \left( \bar{u}^{(2)}_{l,m}(s) + R^{i,(2)}_{l,m}(s) \right) dW_s dW_t. \)

**One has to estimate** \( \mathbb{E}_P \left| \bar{u}^{(3)}_{l,m,n}(r) \right|^2 \) and \( \mathbb{E}_P \left| R^{i,(3)}_{l,m,n}(r) \right|^2 \): the regularity of \( g \) plays a key role.
Fractional regularity: the space $L^{2,\alpha}$

When $E|g(X_T)|^2 < +\infty$, we define

$$V_t, T(g) := E\left|\begin{align*} &g(S_T) - E_F^t P(g(S_T)) \end{align*}\right|^2.$$ 

Definition

For some $\alpha \in (0, 1]$, 

$$L^{2,\alpha} = \{ g \text{ t.q. } \}
\begin{align*} &E(g(S_T))^2 + \sup_{0 \leq t < T} V_t, T(g)(T-t)^\alpha < +\infty \} \}.$$
When $\mathbb{E}|g(X_T)|^2 < +\infty$, we define

$$V_{t,T}(g) := \mathbb{E}_{\mathbb{P}} \left| g(S_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(g(S_T)) \right|^2.$$
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**Definition**

For some $\alpha \in (0, 1]$,

$$L_{2,\alpha} = \left\{ g \text{ t.q. } \mathbb{E}(g(S_T)^2) + \sup_{0 \leq t < T} \frac{V_{t,T}(g)}{(T-t)^\alpha} < +\infty \right\}.$$
Examples

If \( g \) is Lipschitz-continuous, then \( g \in L^{2,1} \).

If \( g \) is Hölder-continuous with exponent \( \alpha \), then \( g \in L^{2,\alpha} \).

If \( g(x) = (x - K)^a + a \) with \( a \in (0,1/2) \), then \( g \in L^{2,a+1/2} \).

If \( g(x) = (x - K)^a + a \) with \( a \in (1/2,1] \), then \( g \in L^{2,1} \).
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- If $g$ is Hölder-continuous with exponent $\alpha$, then $g \in \mathbf{L}_{2,\alpha}$.

- If $g(x) = (x - K)^a$ with $a \in (0, \frac{1}{2})$, then $g \in \mathbf{L}_{2,a+\frac{1}{2}}$.
• If $g$ is Lipschitz-continuous, then $g \in L_{2,1}$.

• If $g$ is Hölder-continuous with exponent $\alpha$, then $g \in L_{2,\alpha}$.

• If $g(x) = (x - K)^a_+$ with $a \in (0, \frac{1}{2})$, then $g \in L_{2,a+\frac{1}{2}}$!

• If $g(x) = (x - K)^a_+$ with $a \in (\frac{1}{2}, 1]$, then $g \in L_{2,1}$!
Examples

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- If $g$ is Hölder-continuous with exponent $\alpha$, then $g \in L_{2,\alpha}$.
- If $g(x) = (x - K)^a$ with $a \in (0, \frac{1}{2})$, then $g \in L_{2,a+\frac{1}{2}}$.
- If $g(x) = (x - K)^a$ with $a \in (\frac{1}{2}, 1]$, then $g \in L_{2,1}$.
- If $g(x) \equiv 1_D(x)$, then $g \in L_{2,\frac{1}{2}}$. 
For $1 \leq l, m, n \leq d$ and $0 \leq t < T$, and using the usual Malliavin representation of Greeks,

\[
\begin{align*}
\mathbb{E}_P \left| \bar{u}_l^{(1)}(t) \right|^2 &\leq C \frac{V_{t,T}(g)}{(T - t)}, \\
\mathbb{E}_P \left| \bar{u}_{l,m}^{(2)}(t) \right|^2 &\leq C \frac{V_{t,T}(g)}{(T - t)^2}, \\
\mathbb{E}_P \left| \bar{u}_{l,m,n}^{(3)}(t) \right|^2 &\leq C \frac{V_{t,T}(g)}{(T - t)^3}. 
\end{align*}
\]
Integrands estimates

-bound for $\mathbb{E}_P \left| \bar{u}_{l,m,n}^{(3)}(t) \right|^2 : \frac{C}{(T-t)^{3-\alpha}}$ if $g \in L_{2,\alpha}$.
Integrands estimates

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- For $R_{l,m,n}^{i,(3)}(t)$, it is more intricate!

$$R_{l,m,n}^{i,(3)}(t) = \cdots - \sum_{0 \leq j < k \leq d} \delta_{t_i}^{j,k} \bar{C}_{l,m,n}^{j,k,(3)}(t) - \cdots$$
Integrands estimates

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$\triangleright$ terms $\frac{\bar{C}_{l,m}^{j,k,(2)}(t)}{\bar{C}_{l,m}^{j,k,(2)}(t_i)}$ et $\frac{\bar{C}_{l,m,n}^{j,k,(3)}(t)}{\bar{C}_{l,m}^{j,k,(2)}(t_i)}$ (with $t_i \leq t \leq t_i+1$)
bound for $\mathbb{E}_P \left| \bar{u}_{l,m,n}^{(3)}(t) \right|^2 : \frac{c}{(T-t)^{3-\alpha}}$ if $g \in L_{2,\alpha}$.

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- using the closed formulas, we obtain that these terms belong to $L_p$ ($p \geq 2$) if and only if $|\pi| \leq \pi^{\text{threshold}}$. 
Integrands estimates

- bound for $\mathbb{E}_p \left| \tilde{u}_{l,m,n}^{(3)}(t) \right|^2 : \frac{C}{(T-t)^{3-\alpha}}$ if $g \in L_{2,\alpha}$.

- For $R_{l,m,n}^{i,(3)}(t)$, it is more intricate!

$$R_{l,m,n}^{i,(3)}(t) = \cdots - \sum_{0 \leq j < k \leq d} \delta_{t_i}^{j,k} \bar{C}_{l,m,n}^{j,k,(3)}(t) - \cdots$$

- terms $\frac{\bar{C}_{l,m}^{j,k,(2)}(t)}{\bar{C}_{l,m}^{j,k,(2)}(t_i)}$ et $\frac{\bar{C}_{l,m,n}^{j,k,(3)}(t)}{\bar{C}_{l,m}^{j,k,(2)}(t_i)}$ (with $t_i \leq t \leq t_{i+1}$)

- using the closed formulas, we obtain that these terms belong to $L_p$ ($p \geq 2$) if and only if $|\pi| \leq \pi^{\text{threshold}}$.

- If $|\pi| \leq \pi^{\text{threshold}}$, then, for $0 \leq t_i \leq t < t_{i+1} \leq T$,

$$\mathbb{E}_p \left| R_{l,m,n}^{i,(3)}(t) \right|^2 \leq \frac{C}{(T-t)^2}.$$
Corollary

Assume $g \in L_{2,\alpha}$ (for some $\alpha \in (0, 1]$) and $\mathbb{E}_P |g(S_T)|^{2p_0} < \infty$ for some $p_0 > 1$. Then, if $|\pi| \leq \pi^{\text{threshold}}$, and for $0 \leq t < T$,

$$\mathbb{E}_P \left| \bar{u}_{l,m,n}^{(3)}(t) + R_{l,m,n}^{i,(3)}(t) \right|^2 \leq \frac{C}{(T-t)^{3-\alpha}}.$$
For some $\beta \in (0, 1]$, 

$$\pi^{(\beta)} := \{ t_k^{(N, \beta)} := T - T \left(1 - \frac{k}{N}\right)^{\frac{1}{\beta}}, 0 \leq k \leq N \}.$$ 

NB.

- $\pi^{(1)} = \text{uniform grid}$.  
- For $\beta < 1$, the points in $\pi^{(\beta)}$ are more concentrated near $T$. 
**Theorem (with uniform grid)**

Assume $g \in L_{2,\alpha}$ and $\mathbb{E}_P |g(S_T)|^{2p_0} < \infty$ for some $p_0 > 1$.

- **Regular grid** $\pi^{(1)}$. For $N$ sufficiently large to ensure $|\pi^{(1)}| = \frac{T}{N} \leq \pi^{\text{threshold}}$, one has

$$(\mathbb{E}_P \left| \bar{E}_{\Delta \Gamma} (g, \pi^{(1)}) \right|^2)^{1/2} = \mathcal{O}\left( \frac{1}{N^{\alpha/2}} \right).$$
Main result

**Theorem (with uniform grid)**

Assume \( g \in L_{2,\alpha} \) and \( \mathbb{E}_P |g(S_T)|^{2p_0} < \infty \) for some \( p_0 > 1 \).

- **Regular grid** \( \pi^{(1)} \). For \( N \) sufficiently large to ensure
  \[ |\pi^{(1)}| = \frac{T}{N} \leq \pi^{\text{threshold}} \], one has

\[
(\mathbb{E}_P |\mathcal{E}_N^\Delta (g, \pi^{(1)})|^2)^{1/2} = O\left(\frac{1}{N^{\alpha/2}}\right).
\]

▶ tight estimate for \( \alpha < 1 \) (if \( \alpha = 1 \), the rate may go from \( N^{1/2} \) to \( N \)).
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Assume $g \in L_{2,\alpha}$ and $\mathbb{E}_P |g(S_T)|^{2p_0} < \infty$ for some $p_0 > 1$.

- **Regular grid** $\pi^{(1)}$. For $N$ sufficiently large to ensure $|\pi^{(1)}| = \frac{T}{N} \leq \pi^{\text{threshold}}$, one has

$$
\left( \mathbb{E}_P \left| \bar{\mathcal{E}}_N (g, \pi^{(1)}) \right|^2 \right)^{1/2} = O\left( \frac{1}{N^{\alpha/2}} \right).
$$

- **tight estimate for** $\alpha < 1$ (if $\alpha = 1$, the rate may go from $N^{1/2}$ to $N$).

- **DGHS with a regular grid does not** improve the rate of convergence.
Main result

Theorem (with non regular grid)

- Non regular grid $\pi^{(\beta)}$, $\beta \in (0, 1)$. For $N$ sufficiently large to ensure $|\pi^{(\beta)}| \leq \pi^{\text{threshold}}$, one has

$$\left( \mathbb{E}_P \left| \mathcal{E}_N^{\Delta \Gamma} (g, \pi^{(\beta)}) \right|^2 \right)^{1/2} = \begin{cases} 
O\left( \frac{1}{N^{2\beta}} \right) & \text{if } \beta \in \left( \frac{\alpha}{2}, 1 \right), \\
O\left( \frac{\sqrt{\log N}}{N} \right) & \text{if } \beta = \frac{\alpha}{2}, \\
O\left( \frac{1}{N} \right) & \text{if } \beta \in \left( 0, \frac{\alpha}{2} \right).
\end{cases}$$

▶ NB. These estimates are equal to those we observe numerically.
Figure: For a Digital Call: at the top (DHS), $\log\left(\mathbb{E}_P|\widehat{\Delta}_N(g, \pi^{(\beta)})|^2\right)$ vs $\log(N)$. At the bottom (DGHS), $\log\left(\mathbb{E}_P|\widehat{\Delta_G}_N(g, \pi^{(\beta)})|^2\right)$ vs $\log(N)$. 
Numerical results

Remark on the convergence in distribution

![Graphs showing tracking errors for Delta hedging and Delta-Gamma hedging]

**Figure:** Distributions of the DHS (at the top) and DGHS (at the bottom) tracking errors for a Digital Call

→ Convergences in $L_2$ and in distribution are different.
Further research

- Extension to more general model for $S$
Further research

- Extension to more general model for $S$
- Rate of convergence in distribution of the DGHS tracking error?