Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle

Holger Kraft\textsuperscript{1}  
Claus Munk\textsuperscript{2}

\textsuperscript{1}Goethe University Frankfurt, Germany

\textsuperscript{2}Aarhus University, Denmark

Bachelier Finance Society
Toronto, June 2010
# Outline

1. **Introduction**
2. **Model**
3. **Solution**
4. **Empirical income**
5. **Robustness**
6. **Summary**
Motivation

- Labor income and housing decisions important for most individuals
- Some papers include labor income, some papers housing decisions
- The few papers including both aspects are restrictive
  [Campbell/Cocco (QJE03), Cocco (RFS05), Yao/Zhang (RFS05), Van Hemert (WP09)]
- Difficult optimization problem – typically solved by highly complex numerical methods
This paper

- Rich model: stochastic labor income, house price, interest rate, stock price
- Disconnect housing consumption and housing investment
- Closed-form “Excel-ready” solution
- Model generates life-cycle behavior with many realistic features
- Non-negligible welfare gains from “perfect” house price-linked financial contracts
Outline

1. Introduction
2. Model
3. Solution
4. Empirical income
5. Robustness
6. Summary
Financial assets

- Short-term interest rate (= return on cash):
  \[ dr_t = \kappa (\bar{r} - r_t) \, dt - \sigma_r \, dW_{rt} \]

- Price \( B_t = B(r_t, t) \) of bond (20Y used later):
  \[ \frac{dB_t}{B_t} = (r_t + \lambda_B \sigma_B(r_t, t)) \, dt + \sigma_B(r_t, t) \, dW_{rt} \]

- Stock price:
  \[ \frac{dS_t}{S_t} = (r_t + \lambda_S \sigma_S) \, dt + \sigma_S \begin{pmatrix} \rho_{SB} & \sqrt{1 - \rho_{SB}^2} \end{pmatrix} \begin{pmatrix} dW_{rt} \\ dW_{St} \end{pmatrix} \]
Housing

“Unit” house price $H_t$: (unit $\approx 1$ “average” sq. foot)

$$\frac{dH_t}{H_t} = \left( r_t + \lambda_H \sigma_H - r^{imp} \right) dt + \sigma_H \left( \rho_{HB}, \hat{\rho}_{HS}, \hat{\rho}_H \right) \begin{pmatrix} dW_{rt} \\ dW_{St} \\ dW_{Ht} \end{pmatrix}$$

Housing positions:

- owning $\varphi_{ot}$ housing units
- renting $\varphi_{rt}$ units at rental rate $\nu H_t$ per unit
- investing in REITs, $\varphi_{Rt}$ units, total return $\frac{dH_t}{H_t} + \nu dt$

Housing consumption: $\varphi_{Ct} = \varphi_{ot} + \varphi_{rt}$

Housing investment: $\varphi_{It} = \varphi_{ot} + \varphi_{Rt}$
Labor income and wealth

Income rate $Y_t$ until retirement at $\tilde{T}$:

$$\frac{dY_t}{Y_t} = (\bar{\mu}_Y(t) + br_t) \, dt + \sigma_Y(t) \left( \rho_{YB}, \hat{\rho}_{YS}, \hat{\rho}_Y \right) \begin{pmatrix} dW_{rt} \\ dW_{St} \\ dW_{Ht} \end{pmatrix}$$

In retirement: $Y_t = \Upsilon Y_{\tilde{T}}, \ t \in [\tilde{T}, T]$.

Human wealth/capital:

$$L_t = E^Q_t \left[ \int_t^T e^{-\int_t^s r_u \, du} Y_s \, ds \right] = \begin{cases} Y_tF(t, r_t), & t < \tilde{T}, \\ Y_{\tilde{T}}F(t, r_t), & t \in (\tilde{T}, T], \end{cases}$$

where $F$ is known in closed form.

Financial/tangible wealth: $X_t$. Total wealth: $X_t + L_t$. 
The individual’s optimization problem

\[
J(t, X, r, H, Y) = \sup E_t \left[ \int_t^T e^{-\delta(u-t)} \frac{1}{1-\gamma} \left( c_u^{\beta} \varphi_{cu}^{1-\beta} \right)^{1-\gamma} ds \right]
\]

Choose:

- \( c_t \): perishable consumption rate
- \( \varphi_c t \): housing units consumed
- \( \hat{\pi}_{lt} \): fraction of total wealth invested in house, \( \hat{\pi}_{lt} = \frac{H_t \varphi_{lt}}{X_t + L_t} \)
- \( \hat{\pi}_{bt} \): fraction of total wealth invested in bond
- \( \hat{\pi}_{st} \): fraction of total wealth invested in stock
## Selected parameter values

<table>
<thead>
<tr>
<th><strong>Individual</strong></th>
<th></th>
<th><strong>Excess stock return</strong></th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wealth</td>
<td>20,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk aversion</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Work life</td>
<td>30 Y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Retirement</td>
<td>20 Y</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>House</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exp. return</td>
<td>1%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility</td>
<td>12%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Imputed rent</td>
<td>5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rent</td>
<td>5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unit price</td>
<td>250</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Income** |  |                           |     |
| Initial     | 20,000                  |                      |     |
| Avg. growth | 2%                      |                      |     |
| Volatility  | 7.5%                    |                      |     |
| Retirement  | 60%                     |                      |     |

| **Correlations** |  |                           |     |
| income/stock,bond | 0 |                      |     |
| house/stock       | 0.5 |                      |     |
| house/bond        | 0.65 |                      |     |
| income/house      | 0.57 |                      |     |
Outline

1. Introduction
2. Model
3. Solution
4. Empirical income
5. Robustness
6. Summary
Solution to the HJB-equation...

\[ J(t, X, r, H, Y) = \frac{1}{1 - \gamma} g(t, r, H)^\gamma (X + YF(t, r))^{1-\gamma}, \]

\[ g(t, r, H) = \frac{\eta \nu}{1 - \beta} H^k \int_t^T e^{-d_1(u-t) - \beta \frac{\gamma-1}{\gamma}} B_\kappa (u-t) r \, du, \]

\[ c = \eta \frac{\beta \nu}{1 - \beta} H^k \frac{X + YF}{g}, \]

\[ \varphi c = \eta H^{k-1} \frac{X + YF}{g}, \]

\[ \hat{\pi}_S, \hat{\pi}_S, \hat{\pi}_S : \text{ see below} \]
Investments – fractions of total wealth

**Stocks**

\[ \hat{\pi}_S = \frac{1}{\gamma \sigma_S} \frac{\xi_S}{\gamma \sigma_S} - \frac{\sigma_Y \zeta_S}{\sigma_S} \frac{L}{X + L}, \]

- 4% \( \leftrightarrow \) 33%

**Bonds**

\[ \hat{\pi}_B = \frac{1}{\gamma \sigma_B} \frac{\xi_B}{\gamma \sigma_B} - \left( \frac{\sigma_Y \zeta_B}{\sigma_B} - \frac{\sigma_r}{\sigma_B} \frac{F_r}{F} \right) \frac{L}{X + L} - \frac{\sigma_r}{\sigma_B} g_r, \]

- 63% \( \leftrightarrow \) 116%

**House**

\[ \hat{\pi}_I = \frac{1}{\gamma \sigma_H} \frac{\xi_I}{\gamma \sigma_H} - \frac{\sigma_Y \zeta_I}{\sigma_H} \frac{L}{X + L} + \frac{H g_H}{g}, \]

- 91% \( \leftrightarrow \) −109%

**Note:** \( \sigma_Y \) drops to zero at retirement, but \( L/(X + L) > 0 \) \( \Leftrightarrow \) jump
Expected wealth over the life-cycle
Expected investments over the life-cycle
... with age-dependent income volatility
Housing consumption and investments
Outline

1. Introduction
2. Model
3. Solution
4. Empirical income
5. Robustness
6. Summary
Empirical income profiles

![Graph showing expected annual income profiles by age and education level. The x-axis represents age in years (25 to 85), and the y-axis represents expected annual income in thousands. The graph includes lines for 'no high', 'high', 'college', and 'constant' education levels, with different line styles for each category.](image-url)
Expected investments again
Housing consumption and investments again

![Graph showing expected housing units over time for different scenarios: Housing cons, no high; Housing inv, no high; Housing cons, high; Housing inv, high; Housing cons, college; Housing inv, college. The graph plots expected housing units against time (years) for different time periods, with peaks and trends indicated for each scenario.]
Outline

1. Introduction
2. Model
3. Solution
4. Empirical income
5. Robustness
6. Summary
Unspanned labor income

- Our solution requires market completeness, i.e., spanned labor income
- Labor income is much closer to being spanned when housing assets are included – high income-house price correlation
- If labor income is unspanned, the implementation of our consumption/investment strategy is sub-optimal
- Bick, Kraft & Munk (presented Thursday): the welfare loss is relatively small (magnitude ≤ 3%)
Constant housing consumption

Note: minimum certainty-equivalent wealth loss is only 0.24%
Infrequent rebalancing of housing positions

<table>
<thead>
<tr>
<th>Adjustment frequency</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 years</td>
</tr>
<tr>
<td>Infrequent $\varphi_C$, frequent $\varphi_I$</td>
<td>0.03%</td>
</tr>
<tr>
<td>Infrequent $\varphi_I$, frequent $\varphi_C$</td>
<td>0.43%</td>
</tr>
<tr>
<td>Infrequent $\varphi_C$ and $\varphi_I$</td>
<td>0.46%</td>
</tr>
</tbody>
</table>

- Suggests moderate welfare gains from market for REITs or CSI housing contracts
- Suggests moderate effects of housing transactions costs
Outline

1. Introduction
2. Model
3. Solution
4. Empirical income
5. Robustness
6. Summary
Summary

- Framework for consumption, housing, and investment decisions over the life-cycle
- High income/house correlation $\leadsto$ life-cycle patterns in optimal decisions, in particular housing investment
- Calibrated model has many realistic features
- Lots of comparative statics in the paper
- Need to know more about typical life-cycle pattern in income volatility and income/house price correlation
- Our model is a benchmark for numerical solutions with portfolio constraints and transaction costs