Interest rate modelling: How important is arbitrage–free evolution?

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Overview

§1 Nelson–Siegel (NS) models:
- Daily yield curve estimation; forecasting.

§2 No arbitrage interest rate models:

§3 Contribution: \[ HJM = NS_{\text{proj}} + \text{Adj}, \text{ Adj is small.} \]
Zero-coupon bonds (ZCB):  
- A ZCB is a contract that guarantees its holder the payment of one unit of currency at time maturity.
- \(P(t, x)\) is the value of the bond at time \(t\) which matures in \(x\) years; \(P(t, 0) = 1\).
- A ZCB price is a discount factor.

Common interest rates:
- Continuously compounded yield: \(y(t, x) = -\frac{\log P(t, x)}{x}\).
- Short rate: \(\lim_{x \to 0^+} y(t, x) = r(t)\).
- Forward rate: \(F(t, x, x + \epsilon) = \frac{1}{P(t+x, \epsilon)} \frac{P(t,x+\epsilon) - P(t,x)}{\epsilon}\).
- Instantaneous forward rate:
  \[f(t, x) = \lim_{\epsilon \to 0^+} F(t, x, x + \epsilon) = -\frac{\partial \log P(t,x)}{\partial x}, \quad r(t) = f(t, t)\]
- Relationship between \(f\) and \(y\): \(y(t, x) = \frac{1}{x} \int_0^x f(t, s) \, ds\).
§1 Nelson–Siegel (NS) models:

- Daily yield curve estimation; forecasting.
Nelson–Siegel Models

No arbitrage models

\[ HJM = NS_{proj} + \text{Adj}, \text{Adj small} \]

Yield curve estimation.

Figure: The EUR ZERO DEPO/SWAP curve as of 24/06/2009.
Yield curve estimation.

**Figure:** The EUR ZERO DEPO/SWAP curve as of 24/06/2009.
Yield curve estimation.

- Nelson–Siegel curves (and their extensions) are used by banks (e.g., central/investment) to estimate the shape of the yield curve.
- This estimation is justified by principal component analysis: low number of dimensions describes the curve with high accuracy.

Nelson–Siegel yield curve:

\[ y(x) = L + S \left( \frac{1 - e^{-\lambda x}}{\lambda x} \right) + C \left( \frac{1 - e^{-\lambda x}}{\lambda x} - e^{-\lambda x} \right) \]

- \( y \) denotes the Nelson–Siegel yield curve.
- \( \lambda, L, S \) and \( C \) are estimated using yield data.
**Nelson–Siegel Models**

**Yield curve estimation.**

**Figure:** Influence of shocks on the factor loadings of the Nelson–Siegel yield curve.
Forecasting the term structure of interest rates.

Nelson–Siegel yield curve forecasting model:

\[ y(t, x) = L(t) + S(t) \left( \frac{1 - e^{-\lambda x}}{\lambda x} \right) + C(t) \left( \frac{1 - e^{-\lambda x}}{\lambda x} - e^{-\lambda x} \right) \]

Advantages:

- Simple implementation.
- Easy to interpret.
- Can replicate observed yield curve shapes.
- Can produce more accurate one year forecasts than competitor models (Diebold and Li 2007).

A drawback?

Nelson–Siegel models are not arbitrage–free (Filipović 1999).
§2 No arbitrage interest rate models:

The HJM framework:

\[ df(t, x) = \alpha(f, t, x) \, dt + \sigma(f, t, x) \, dW(t), \]
\[ f(0, x) = f^0(x), \]

where

\[ \alpha(f, t, x) = \frac{\partial f(t, x)}{\partial x} + \sigma(f, t, x) \int_0^x \sigma(f, t, s) \, ds \]

A concrete model is fully specified once \( f^0 \) and \( \sigma \) are given.
The HJM framework

Why use the HJM framework?
- Most short rate models can be derived within this framework.
- Automatic calibration: initial curve is a model input.
- Arbitrage-free pricing.

Interesting points:
- In practice one uses 2–3 driving Brownian motions ("factors").
- Despite this most HJM models are infinite–dimensional.
- Choice of volatility (not number of factors) determines complexity.
- A HJM model will be finite–dimensional if the volatility is an exponential polynomial function ie

$$EP(x) = \sum_{i=1}^{n} p_{\lambda_i}(x) e^{-\lambda_i x},$$

where $p_{\lambda_i}$ is a polynomial associated with $\lambda_i$, (Björk, 2003).
The HJM framework

\[ df(t, x) = \alpha(f, t, x) \, dt + \sigma(f, t, x) \, dW(t), \]
\[ f(0, x) = f^0(0, x). \]

Possible volatility choices:

- **Hull–White**: \( \sigma(f, t, x) = \sigma e^{-ax} \), (Ho–Lee: \( \sigma(f, t, x) = \sigma \)).
- **Nelson–Siegel**: \( \sigma(f, t, x) = a + (b + cx)e^{-dx} \).
- **Curve–dependent**: \( \sigma(f, t, x) = f(t, x)[a + (b + cx)e^{-dx}] \).

**Note**: Curve–dependent volatility is similar to a continuous–time version of the BGM/LIBOR market model.
§3a Theoretical Contribution: $HJM = NS + Adj = NS_{proj} + Adj$, Adj small
A specific HJM model

Consider the following HJM model:

\[ df(t, x) = \left( \frac{\partial f}{\partial x} + C(\sigma, x) \right) \, dt + \sigma_{11} \, dB_1(t) \]
\[ + (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} xe^{-\lambda x}) \, dB_2(t), \]
\[ f^0(0, x) = f^{NS}(x). \]
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(Björk 2003): \( f \) has a finite dimensional representation (FDR) since there is a finite-dimensional manifold \( \mathcal{G} \) such that

- \( f^o \in \mathcal{G} \)
- drift and volatility are in the tangent space of \( \mathcal{G} \).
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- \( f^0 \in \mathcal{G} \)
- drift and volatility are in the tangent space of \( \mathcal{G} \).

For our model \( \mathcal{G} = \text{span}\{\overline{B}(x)\} \)

- \( \overline{B}(x) = (1, e^{-\lambda x}, xe^{-\lambda x}, x, x^2 e^{-\lambda x}, e^{-2\lambda x}, xe^{-2\lambda x}, x^2 e^{-2\lambda x}) \)
A method to construct the FDR

- $f$ has an FDR given by $f(t, x) = \tilde{B}(x).z(t)$ where
  
  $$dz(t) = (Az(t) + b) \, dt + \Sigma \, dW(t), \quad z(0) = z_0.$$  

- $A$, $b$, $\Sigma$ and $z_0$ are determined from
  
  - $\tilde{B}(x).z_0 = f^0(x)$
  - $\tilde{B}(x).b = C(\sigma, x) = (B(x)\sigma) \int_0^x (B(s)\sigma)^T \, ds$
  - $\tilde{B}(x)Az(t) = \frac{\partial f}{\partial x} = \frac{d\tilde{B}(x)}{dx} z(t)$
  - $\tilde{B}(x)\Sigma = B(x)\sigma$

- $B(x) = (1, e^{-\lambda x}, xe^{-\lambda x})$

- Easily generalised to exponential–polynomial functions.
Our specific HJM model

Our specific HJM model has the following finite–dimensional representation:

\[ f(t, x) = z_1(t) + z_2(t) e^{-\lambda x} + z_3(t) xe^{-\lambda x} + z_4(t)x \]
\[ + z_5(t)x^2 e^{-\lambda x} + z_6(t)e^{-2\lambda x} + z_7(t)xe^{-\lambda x} + z_8(t)x^2 e^{-2\lambda x}. \]

**Interesting points:**

- *Only* \( z_1, z_2 \) and \( z_3 \) are stochastic.
- A *specific* choice of initial curve will result in \( z_4, \ldots, z_8 \) being constant.
  (This is closely related with work by Christensen, Diebold and Rudebusch (2007) on extended NS curves).
- This model has counter–intuitive terms.
Our specific HJM model

How important is the Adjustment in the HJM model?

Previous approach: Statistical
- Coroneo, Nyholm, Vidova–Koleva (ECB working paper 2007).
- The estimated parameters of a NS model are not statistically different from those of an arbitrage–free model.

Our approach: Analytical
- We quantify the distance between forward curves,
- We analyse the differences in interest rate derivative prices.
**Our Nelson-Siegel model: \( \text{NS}_{\text{proj}} \)**

- \( \text{NS}_{\text{proj}}(t, x) = \hat{z}_1(t) + \hat{z}_2(t)e^{-\lambda x} + \hat{z}_3(t)xe^{-\lambda x} \)

- \( \hat{z}(t) = (\hat{A}\hat{z}(t) + \hat{b}) dt + \hat{\Sigma} dW(t), \ z(0) = z_0 \) where
  - \( B(x)\hat{z}_0 = f^{NS}(x) \)
  - \( B(x)\hat{b} = \mathcal{P}[(B(x)\sigma) \int_0^x (B(s)\sigma)^T \, ds] \)
  - \( B(x)\hat{A}\hat{z}(t) = \frac{\partial f}{\partial x}^{\text{NS}_{\text{proj}}} \)
  - \( B(x)\hat{\Sigma} = \mathcal{P}[B(x)\sigma] \)

**Projection formula:** Projection of \( v \) onto \( \text{Span} \ (B_1, B_2, B_3) \):

\[
\mathcal{P} : L^2 \rightarrow \text{Span} \ B(x) : v \mapsto \sum_{i=1}^{3} \sum_{j=1}^{3} (R^{-1})_{ij} < v, B_j > B_i(x),
\]

\[
R_{ij} = \int B_i(s)B_j(s) \, ds
\]
Our Nelson-Siegel model: $\text{NS}_{\text{proj}}$

- $\text{NS}_{\text{proj}}(t, x) = \hat{z}_1(t) + \hat{z}_2(t)e^{-\lambda x} + \hat{z}_3(t)xe^{-\lambda x}$

- $\hat{z}(t) = (\hat{A}\hat{z}(t) + \hat{b}) dt + \hat{\Sigma} dW(t), z(0) = z_0$ where
  - $B(x).\hat{z}_0 = f^{\text{NS}}(x)$
  - $B(x).\hat{b} = \mathcal{P}[(B(x)\sigma) \int_0^x (B(s)\sigma)^T ds]$
  - $B(x)\hat{A}\hat{z}(t) = \frac{\partial f}{\partial x}^{\text{NS}_{\text{proj}}}$
  - $B(x)\hat{\Sigma} = \mathcal{P}[B(x)\sigma]$

$HJM = \text{NS}_{\text{proj}} + \text{Adj} = \text{NS}_{\text{proj}} + \text{Adj}$

$\text{Adj} < \text{Adj}$

- Same approach can be used for infinite-dimensional HJM.
§3b Applied Contribution: $HJM = NS + Adj = NS_{proj} + Adj$

$Adj < Adj$, $Adj$ is small.
Recall the HJM model:

\[
\frac{df(t, x)}{dt} = \left( \frac{\partial f}{\partial x} + C(\sigma, x) \right) dt + \sigma_{11} dB_1(t) \\
+ \left( \sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} xe^{-\lambda x} \right) dB_2(t),
\]

\[
f^0(0, x) = f^{NS}(x).
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\[ + \left( \sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x} \right) dB_2(t), \]
\[ f^0(0, x) = f^{NS}(x). \]

We can rewrite this model as:

\[ dY(t, x) = \mu(t, x) \ dt + S_1(x) \ dB_1(s) + S_2(x) \ dB_2(s), \]

where \( Y(t, x) = \log P(t, x), \)
\[ S_1(x) = \sigma_{11} x, \]
\[ S_2(x) = \frac{e^{-x\lambda}(1+e^{x\lambda})(\lambda\sigma_{22} + \sigma_{23})}{\lambda^2} + x \left( \sigma_{21} - \frac{e^{-x\lambda}\sigma_{23}}{\lambda} \right). \]
HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.
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To estimate the volatility we use **Principal Component Analysis (PCA)**.
HJM parameter estimation

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\[
\begin{pmatrix}
\Delta Y(t, 1) \\
\vdots \\
\Delta Y(t, 20)
\end{pmatrix}
\approx
\begin{pmatrix}
\mu_1(1) \\
\vdots \\
\mu_1(20)
\end{pmatrix}
+ \begin{pmatrix}
S_1(1) & S_2(1) \\
S_1(20) & S_2(20)
\end{pmatrix} \ast \begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\]
HJM parameter estimation

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\vdots \\
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\vdots & \vdots \\
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Z_2
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By applying PCA to our data set

- we found that approximately 98% of the variance in the yields is captured by the **first two principal components**.
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\begin{pmatrix}
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\vdots & \vdots \\
S_1(20) & S_2(20)
\end{pmatrix}
\ast
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\]

By applying PCA to our data set
- we found that approximately 98% of the variance in the yields is captured by the **first two principal components.**
- we determined the volatility associated with each factor.
No arbitrage models

\[ HJM = NS_{proj} + \text{Adj}, \text{ Adj small} \]

Parameter estimation

**Figure:** First and second principal component and fitted curves.

- **First component fitted using** \( S_1(x) = \sigma_{11} x \).
- **Second component fitted using**
  \[
  S_2(x) = \frac{e^{-x\lambda} \left( -1 + e^{x\lambda} \right)}{\lambda^2} \left( \lambda \sigma_{22} + \sigma_{23} \right) + x \left( \sigma_{21} - \frac{e^{-x\lambda} \sigma_{23}}{\lambda} \right).
  \]
Graphical analysis

**Figure:** Some possible curve shapes generated by HJM and NS models after simulation for 5 years.
Nelson–Siegel Models

No arbitrage models

\[ HJM = NS_{proj} + \text{Adj}, \text{ Adj small} \]

Graphical analysis

**Figure:** Some possible curve shapes generated by HJM and NS models after simulation for 5 years.
**Graphical analysis**

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**Graphical analysis**

**Figure:** Some possible curve shapes generated by HJM and NS models after simulation for 5 years.

**Note:** The average curve for any future time can be calculated analytically at time 0.
Figure: Difference in the curves after five years.

Note: This difference remains the same for each realisation.
### Analysis of simulated prices

#### Theoretical ‘European call option’ prices on a 20 year bond:

<table>
<thead>
<tr>
<th>$T_0$ (years)</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strike</strong></td>
<td>0.565</td>
<td>0.686</td>
<td>0.865</td>
</tr>
<tr>
<td>$\Pi_{HJM}(T_0)$</td>
<td>0.0193</td>
<td>0.0146</td>
<td>0.00567</td>
</tr>
<tr>
<td>$\Pi_{NS_{proj}}(T_0)$</td>
<td>0.0192</td>
<td>0.0144</td>
<td>0.00562</td>
</tr>
<tr>
<td><strong>% difference</strong></td>
<td>0.47%</td>
<td>1.18%</td>
<td>0.89%</td>
</tr>
</tbody>
</table>

- $T_0$ denotes option maturity; $\Pi$ denotes price.
- The strike is the at–the–money forward price of the bond $P(T_0, 20 - T_0)$. 

$HJM = NS_{proj} + \text{Adj, Adj small}$
### Analysis of simulated prices

<table>
<thead>
<tr>
<th>Cap</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_{HJM}$</td>
<td>0.372</td>
<td>0.349</td>
<td>0.256</td>
</tr>
<tr>
<td>$\Pi_{NS_{proj}}$</td>
<td>0.371</td>
<td>0.256</td>
<td>0.163</td>
</tr>
</tbody>
</table>

| % difference (of nominal) | 0.045% | 0.043% | 0.033% |

- Maturity of 20 years; nominal of 1; annual interest rate payment.
- Differences of 1–2% of nominal are common.
Case Study 1: Cap/Floor

- Nominal: EUR 180 million; Maturity: 30/6/2014.
- Receive capped and floored 3 month EURIBOR + spread:
  \[
  \text{Payout} = \left(\frac{\text{Nominal}}{4}\right) \times \max[0, \min[5\%, \text{ir} + 0.3875%]]
  \]

Valuation:

<table>
<thead>
<tr>
<th>Model</th>
<th>Valuation (EUR)</th>
<th>% difference (of nominal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerix (1F HW)</td>
<td>$-2,045,140$</td>
<td>0.022%</td>
</tr>
<tr>
<td>HJM</td>
<td>$-2,085,124$</td>
<td>0.022%</td>
</tr>
<tr>
<td>NS_{proj}</td>
<td>$-2,085,449$</td>
<td>0.024%</td>
</tr>
</tbody>
</table>
Case Study 2: Curve Steepener

- Nominal: EUR 4,258,000; Maturity: 30/5/2015.
- Pay ‘Curve steepener payoff’ semi–annually:

  Payout = (Nominal/2) * \( \text{Max["10 year swap"-"2 year swap",0]} \)

- Valuation:

<table>
<thead>
<tr>
<th>Model</th>
<th>Valuation (EUR)</th>
<th>% difference (of nominal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerix (3F BGM)</td>
<td>345,186</td>
<td></td>
</tr>
<tr>
<td>HJM</td>
<td>295,401</td>
<td>1.2%</td>
</tr>
<tr>
<td>(NS_{proj})</td>
<td>296,251</td>
<td>1.15%</td>
</tr>
</tbody>
</table>
Contribution

1. \( HJM = NS + Adj \)
   - Initial curve affects shape of \( Adj \), \( Adj \) contains counter-intuitive terms.

2. \( HJM = NS_{proj} + Adj, \quad Adj < Adj. \)

3 Simulation and Case Studies: \( HJM \approx NS_{proj} \)
   - For forward curve shapes, bond options, capped FRNs.
   - Numerix (3F–BGM) \( \approx 2F–HJM \approx NS_{proj} \)
Nelson–Siegel Models

No arbitrage models

$HJM = NS_{proj} + Adj$, $Adj$ small

Thank you

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