Ambiguity Aversion in Real Options:

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The Real Option Problem

- Classical work of **McDonald & Siegel (86)** assigns the value

\[ f_t = \mathbb{E}_t \left[ e^{-\rho(T-t)} \left( P_T - I \right)_+ \right] \]

...to the option to invest in a project at \( T \)
- \( P_t \) – value of a project if invested in at time \( t \)
- \( I \) – the cost of the investment
- \( \rho \) – discount rate
- If **early investment** is allowed (e.g. qrtly or mthly), then

\[ f_t = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t \left[ e^{-\rho(\tau-t)} \left( P_\tau - I \right)_+ \right] \]

- \( \mathcal{T} \) – a set of admissible stopping times
The Real Option Problem

- $P_t$ often assumed **spanned** by a traded asset – mostly **unrealistic**
  - Spanning allows the project to effectively be traded and therefore valued using discounted expectations
- Instead view $P_t$ as **strongly correlated** to a **tradable asset** $S_t$
- Two key questions addressed here:
  - How to value the option on $P_t$ by trading in $S_t$?
    - Will use **Utility indifference pricing**
    - Henderson & Hobson (02) and Henderson (07) for perpetual version
  - An agent may have a good model for $S_t$ but not $P_t$… how to account for this **ambiguity**?
    - **Knightian Uncertainty / ambiguity aversion**
    - **Robustness Approach:** Anderson, Hansen, & Sargent (99); Uppal & Wang (03); Maenhout (04); and J. & Sigloch (09)
    - **Recursive multiple priors:** Epstein & Wang (94) Chen & Epstein (02) extension of Gilboa & Schmeidler (89)
Utility Indifference Pricing

- Consider:
  - Suppose want to value the risk $Y$ received at $T$
  - Agent's utility is exponential $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$
  - Agent's initial wealth is $x$ and risk-free rate is $r$

- Basic utility indifference valuation:
  1. Invest all of $x$ in bank account:
     \[ V(x) = -\frac{1}{\gamma} e^{-\gamma x} e^{rT} \]
  2. Invest $x - v$ in bank account and receive $Y$ at $T$:
     \[ U(x) = \mathbb{E}[u((x - v)e^{rT} + Y)] = V(x - v)\mathbb{E}[e^{-\gamma Y}] \]
  3. Indifference value $v$ solves
     \[ V(x) = U(x) \Rightarrow v = -\frac{1}{\gamma} e^{-rT} \ln \mathbb{E}[e^{-\gamma Y}] \]
Utility Indifference Pricing

- Invest optimally in $S_t$ **without option** to invest in project

$$U(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[u(X_T)]$$

  - classical Merton (69) problem, admits explicit solution

- Invest optimally in $S_t$ **with option** to invest in project
  - Upon exercise, receive option value, and revert to Merton:

$$U(x, P; a) = \sup_{\tau \in \mathcal{T}} \sup_{\pi \in \mathcal{A}} \mathbb{E}[V(\tau, X_\tau + a(P_\tau - I)_+) ]$$

$$V(t, x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[u(X_T)|X_t = x]$$

  - Henderson (07) solved the perpetual version of this problem
Indifference value \( v \) of option to invest in project defined as

\[
U(x, P; 0) = U(x - v, P; 1)
\]
Non-traded project value $P_t$ and correlated traded equity $S_t$ satisfy

$$dP_t = P_t \left( \nu dt + \eta dW_t^P \right), \quad dS_t = S_t \left( \mu dt + \sigma dW_t^S \right)$$

with $d[W^P, W^S]_t = \rho dt$.

For risk-neutral valuation can use the **minimal entropy martingale measure**:

$$dP_t = P_t \left( \hat{\nu} dt + \eta d\hat{W}_t^P \right), \quad dS_t = S_t \left( r dt + \sigma d\hat{W}_t^S \right)$$

with $\hat{\nu} = \nu - \rho \eta \frac{\mu - r}{\sigma}$ and $d[\hat{W}^P, \hat{W}^S]_t = \rho dt$

The **MEMM** appears in indifference valuation as well

**Ambiguity adjusted MEMM** appears for ambiguity-averse agents
Utility Indifference Pricing

- Let $X_t$ denote the investor’s wealth
- Let $\pi_t$ denote the dollar amount invested in the tradable asset $S_t$
- Let $\mathcal{A}$ denote the set of admissible strategies

$$\mathcal{A} = \left\{ \pi_t \mid \text{self financing and} \quad \int_0^T \pi_t^2 \, ds < +\infty \right\}$$

- Self-financing strategies imply

$$dX_t = \left( (\mu - r)\pi_t + r X_t \right) dt + \sigma \pi_t \, dW_t^S$$
Dynamic programming principle leads to the HJB eqn

\[
\begin{aligned}
\partial_t U + \max_{\pi} \mathcal{L}_\pi U &= 0 \\
U(t, b(x), P; a) &= V(t, x + a(P - I)_+) 
\end{aligned}
\]
Utility Indifference Pricing

- Assume exp. utility: $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$ then wealth factors:

$$V(t, x) = u \left( x e^{r(T-t)} \right) e^{-\frac{1}{2} \lambda^2 (T-t)}$$

$$U(t, x, e^y) = V(t, x) G^\beta(t, y)$$

where $\lambda = (\mu - r)/\sigma$ is the market price of risk
and $\beta = (1 - \rho^2)^{-1}$ is the power transform coefficient

- $G$ solves a linear complementarity problem

$$\begin{cases} 
\partial_t G + \mathcal{L} G & \leq 0, \\
\ln G(t, y) & \geq h(t, y), \\
(\partial_t G + \mathcal{L} G) \cdot (\ln G(t, y) - h(t, y)) & = 0,
\end{cases}$$

where

$$h(t, y) = a \frac{\gamma}{\beta} (e^y - K)_+ e^{r(T-t)}, \quad \text{and,} \quad \mathcal{L} = \hat{\nu} \partial_y + \frac{1}{2} \eta^2 \partial_{yy}$$
Since wealth factors, the **indifference value** is simply:

\[
v(t, y) = \frac{\beta}{\gamma} e^{-r(T-t)} \ln G(t, y)
\]

\(v(t, y)\) then satisfies a **non-linear complementarity problem**:

\[
\begin{align*}
\partial_t v + \mathcal{L} v - \frac{1}{2} \eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_y v)^2 & \leq r v, \\
(\partial_t v + \mathcal{L} v - \frac{1}{2} \eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_y v)^2 - r v) \\
\cdot (v(t, y) - (e^y - K)_+) & = 0.
\end{align*}
\]

As \(\gamma \downarrow 0\), the non-linearity disappears

Recovers the risk-neutral American option price
The effect of **risk-aversion** on **exercise policy**
Utility Indifference Pricing

The effect of risk-aversion on option value

![Graph showing the relationship between risk-aversion and option value.](image)
Agent's may lack confidence in their model and this uncertainty affects decisions.

As illustrated in the classical Ellsberg paradox:

- You are given 40 red marbles; and a total of 60 black and green marbles.
- Mix all marbles, 1 chosen at random.
- Most investors prefer A to B:
  - A: receive $100 if red
  - B: receive $100 if black

- Most investors prefer D to C:
  - C: receive $100 if red or green
  - D: receive $100 if black or green

Inconsistent with maximizing expected utility.
Resolved through including ambiguity aversion.
Agent’s may **lack confidence** in their model
- Knightian Uncertainty viewed as **ambiguity aversion**
- Use ideas from **Robust Portfolio Optimization**
  - Agent has some confidence in a **reference measure** $\mathbb{P}$
  - Agent is willing to consider a class of **candidate measures** $\mathbb{Q}$
  - Agent then solves the problem

$$V(x, P, S) = \sup_{\pi \in A} \inf_{Q \in Q} \mathbb{E}_x^Q_{\pi,P,S} \left[ u(X_T^\pi) + \frac{1}{\varepsilon} h(Q|\mathbb{P}) \right].$$

- $h(Q|\mathbb{P})$ is a **penalty function**... e.g. relative entropy
- The parameter $\varepsilon$ acts as a measure of ambiguity aversion
  - As $\varepsilon \downarrow 0$ reference measure is picked out
  - $\varepsilon \uparrow +\infty$ all candidates measures are equal
Robust Utility Indifference

- For relative entropy: \( h(Q|P) = \mathbb{E}^{Q}[\ln \frac{dQ}{dP}] = \mathbb{E}^{Q}[\int_0^T \mu'_s \Sigma^{-1} \mu_s ds] \)

- Instead use scaled relative entropy similar to in J. & Sigloch (09):

\[
U^a(t, x, P, S) = \sup_{\tau \in \mathcal{T}_t} \sup_{\pi \in \mathcal{A}} \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ V(\hat{\tau}, X_{\hat{\tau}}^\pi + a(P_{\hat{\tau}} - I)_+, P_{\hat{\tau}}, S_{\hat{\tau}}) - \frac{1}{\epsilon} \int_0^{\hat{\tau}} U^a(s, X_s^\pi, P_s, S_s) \mu'_s \Sigma^{-1} \mu_s ds \right],
\]

where, \( \hat{\tau} = \tau \land T \) and

\[
V(t, x, P, S) = \sup_{\pi \in \mathcal{A}} \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ u(X_T^\pi) - \frac{1}{\epsilon} \int_t^T V(s, X_s^\pi, P_s, S_s) v_s^Q \Sigma^{-1} \mu_s^Q ds \right].
\]
Robust Utility Indifference

- The **Dynamic programming principle** leads to the HJB eqn

\[
\begin{align*}
\partial_t U + \max_{\pi, \mu} \left( \mathcal{L}_{\pi, \mu} U - \frac{1}{\varepsilon} \mu' \Sigma^{-1} \mu U \right) &= 0 \\
U(t, b(x), P; a) &= V(t, x + a(P - l)_+) \\
\partial_t V + \max_{\pi, \mu} \left( \mathcal{L}_{\pi, \mu} V - \frac{1}{\varepsilon} \mu' \Sigma^{-1} \mu V \right) &= 0 \\
V(T, x) &= u(x)
\end{align*}
\]

- The scaling of relative entropy allows explicit solutions the DPE
- Equations are similar to previous case with modified parameters
Robust Utility Indifference

- The ansatz

\[ V(t, x) = u \left( x e^{r(T-t)} \right) e^{-\frac{1}{2} \lambda^2 (T-t)}, \quad U(t, x, e^y) = V(t, x) G^\beta(t, y) \]

solves the resulting dynamic programming equations

- \( \lambda^2 = \frac{1}{1+\varepsilon} \left( \frac{\mu-r}{\sigma} \right) \) is **ambiguity adjusted market price of risk**

- The power transform coefficient \( \beta \) also depends on the ambiguity aversion parameter

- **Indifference value** \( v(t, y) = \frac{\beta}{\gamma} e^{r(T-t)} \ln G(t, y) \) solves a non-linear complimentary problem

\[
\begin{aligned}
\partial_t v + \mathcal{L}_\varepsilon v - \frac{1}{2} \eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_y v)^2 & \leq r v, \\
v(t, y) & \geq (e^y - K)_+, \\
\left( \partial_t v + \mathcal{L}_\varepsilon v - \frac{1}{2} \eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_y v)^2 - r v \right) \\
\cdot (v(t, y) - (e^y - K)_+) & = 0.
\end{aligned}
\]
The effect of **ambiguity-aversion** on **exercise boundary**
Robust Utility Indifference

The effect of **ambiguity-aversion** on **option price**

![Graph showing the effect of ambiguity-aversion on option price](image-url)
Robust Utility Indifference

- **Ambiguity and Risk aversion** are similar but distinct
- As $\gamma \downarrow 0$ non-linearity in LC problem is removed but dependence on $\varepsilon$ remains through the *ambiguity adjusted MEMM drift*

$$\hat{\nu} = \nu - \frac{1}{1 + \varepsilon \rho \eta} \frac{\mu - r}{\sigma}$$

- As $\varepsilon \downarrow 0$, $\hat{\nu}$ decreases to **MEMM drift**
- As $\varepsilon \uparrow +\infty$, $\hat{\nu}$ increases to $\nu - \text{reference measure drift}$

- An agent may be risk-neutral but severely ambiguity averse
Conclusions

- Project value modeled as non-traded asset
- Correlated traded asset provides partial hedge
- Use utility indifference to value option
- Risk-aversion affects option value and exercise strategy in non-linear way
- Ambiguity aversion can be incorporated through a scaled entropic penalty
- Ambiguity also affects option value and exercise strategy in non-linear way
- Ambiguity and risk aversion are similar but distinct factors in explaining agent's behavior
Thank you for your attention!!

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