Economic Default vs. Default

Adrien de Larrard

UC Berkeley, Ecole Normale Superieure, Universite Paris VI.
Based on joint work with Xin Guo from UC Berkeley and Robert Jarrow from Cornell University

June 25, 2010
1 Default, a brief review

2 Why economic default?

3 Model economic default by equity-debt structure
   - The model
   - Analytical results
     - Distribution of the real default
     - Distribution of economic default
   - Examples

4 Conclusions
In 1974, Merton introduced the first structural model using the Black and Scholes theory on option pricing.

In 1976, Black and Cox introduced the first credit risk model using the first hitting time below a barrier; This barrier can be set exogenously, or endogenously to satisfy the shareholders.

Many people tried to improve structural models, adding jumps at the diffusion for instance.

However, most of the time short term credit spreads are underestimated.
Modeling default: intensity-based models

- These models do not consider the relation between default and firm value in an explicit manner.
- Default comes as a 'surprise'.
- There is an intensity of the arrival of default $\lambda_t$.
- $\lambda_t$ is an exogenous random process.
- Default can be not predicted: totally inaccessible.
Modeling default: latest models

- Noisy partial information: (Duffie and Lando, 2001) The information (filtration) is not the same for the investors and for the firm managers. Hence the spread in strictly positive when $T \to 0$.

- Delayed information models (Schonbucher 2003, Guo Jarrow and Zeng 2007, Collin-Dufresne et al. 2002)

- Information-based models are a mixture of the structural models and the reduced-form models.
Reality check

Recently, an empirical study conducted by Guo, Jarrow and Lin on 2500 defaulted bonds showed that:

- The market anticipates the default before it actually occurs.
- They proposed a new definition: The **Economic default** as the first time the market prices the debt as if it defaulted.
- Their statistical definition:

\[
\tau_e = \inf\left\{ t : B_t \leq B^d_T e^{-\int_t^\tau r_s ds} \right\},
\]

given \( \tau \), where \( \tau \) is the real default.

- Using economic default, the average pricing error between economic default and recorded default is less than one basis point.
Figure 1: Delta Airlines Inc., coupon 8.3%, maturity 12/15/2029. Time series graph of debt prices as a percentage of face value ($100). The solid vertical line represents the default date.
Economic default

Figure 5: Histogram of the Time Between the Economic and Recorded Default Dates. The number of nonzero differences is 73.
Debt and equity structure of the firm

- In our model, the firm needs to pay back its debt at discrete times $T_1, T_2, \ldots, T_n$ where for all $k \geq 1$, $T_k = kT$ for some $T > 0$ ( $T$ being the time between two consecutive pay back time for instance: 1 week, 3 months....).

- For each of these times, the amount of debt that has to be paid pack is $D_k$.

- To simplify we assume that $D_{k+1} - D_k$ is constant over time. This is the case when the firm is rolling over its debt with a constant spread.

- The value of the firm $(S_t, t \geq 0)$ follows a geometric Levy process.
Default and economic default

As for structural models, we set for all time $T_k$ a threshold $B_k$ function of the whole sequence $(D_k, k \geq 0)$. For instance:

- $B_k = D_k$ if we only focus on the short term debt.
- $B_k$ can be the whole amount of debt. This is the threshold for the classic Merton model.
- Similar to the KMV model, $B_k$ can be equal to $D_k$ (short term debt) + a fraction of long term debt.
In our model, the default occurs when the asset value of the firm is less that the threshold $B_k$ at a time $T_k$. The default can only happen for some time $T_k$:

$$\tau = \inf \{ T_k, V_{T_k} < B_k \}.$$ 

Mathematically, the economic default is defined as the last time the asset value of the firm is above this threshold:

$$\tau_e = \sup \{ t > T_{\tau-1}, V_t \geq B_{\tau} \}.$$
Our model
Our model
The law of the real default $\tau$?

For all $n \geq 0$, $x \geq 0$ we want to compute

$$u_n(x) = \mathbb{P}[\tau = T_n| S_0 = x, T, B_1, B_2, ...].$$

- If for all $k \geq 0$, $B_k = B$ is constant (we have the same threshold for all times $T_k$), $\tau$ is the first hitting time under $\log(B)$ of the random walk ($M_i = \log(S_{iT})$, $i \geq 0$).
- If $(B_{k+1} - B_k, k \geq 1)$ is constant, $\tau$ is the first hitting time under $\log(B)$ of the random walk ($M_i = \log(S_{iT}) - (B_{k+1} - B(k))/T$, $i \geq 0$).
Law of the real default $\tau$

Using the fluctuation theory of random walks we prove that:

**Theorem**

\[
\mathbb{P}[\tau = k, S_\tau \in du|S_0 = x, T, B_1, B_2, ...] = \\
\int_0^\infty \int_0^y \sum_{i=0}^{k-1} U^-(\log(x/B) - dy, k - i)U^+(dv - y, i)F(-v - du),
\]

- where $U^+$ (resp $U^-$) is the green function of the increasing (resp decreasing) ladder process (cf: fluctuation theory) of the random walk $(M_i, i \geq 0)$.
- $F$ is the cdf of $M_1$. 
Property of the credit spread

- Using the same method we can compute
  \[ u_n(t, x) = \mathbb{P}[\tau = T_n | S_t = x, T, B_1, B_2, ...]. \]

- The credit spread is a function of both the equity dynamics and the debt structure.

- Depending on the dynamics of the asset value \((S_t, t \geq 0)\), and on the tresholds \((B_k, k \geq 1)\), we can generate increasing, decreasing or bumped credit spreads.

- For a general debt structure \(((T_k, D_{T_k}), k \geq 1)\), we can’t use the fluctuation theory of random walks. We have to do Monte Carlo simulations to compute the sequence \((u_n(x, t), n \geq 1)\).
Technical difficulties:

- This is not a stopping time.
- This is a last passage time and not a first passage time.
- It is defined conditioned on the knowledge of the real default $\tau$.

Main idea

We first compute the distribution of the time between the economic default and the real default $\tau - \tau_e$ conditioned on the default time $\tau = T_n$ and then get the unconditioned law by summing over all $T_n$ and using the Markov property of Levy processes.
The law of $\tau_e$ conditioned on $\tau$

When the dynamics of the asset value $(S_t, t \geq 0)$ is continuous:

$$
\mathbb{P}[\tau - \tau_e \in ds | S_{T_{k-1}} = x, \tau = T_k] = \int_0^T \frac{A_u(s)}{\mathbb{P}[H_{B-x} \leq T]} \phi(u) du.
$$

- $H_{B-x}$ is the first hitting time under $B - x$ of a stochastic process with law $(S_t, t \geq 0)$ starting from 0. $\phi(u)$ is its pdf.
- $A_T$ is the pdf of the last passage at zero time before $T$ of a stochastic process with law $(S_t, t \geq 0)$ starting from 0. This law is the arcsine law if $(S_t, t \geq 0)$ is a Brownian motion.
Arcsine law of $\tau - \tau_e$

- The unconditioned law of $\tau_e - \tau$ is obtained by summing over all $n \geq 1$.
- Both conditioned and unconditioned distribution of the time between economic default and real default follow a mixture of arcsine law (U-shaped).

- We conjecture that this is still true for jump-diffusion Levy processes.
Figure 5: Histogram of the Time Between the Economic and Recorded Default Dates. The number of nonzero differences is 73.
Example 1

Here the dynamics of the firm is a geometric Brownian motion, $\sigma = 0.25$, $r = 0.04$. Leverage = 0.8. $T = 15$ days. $B_1 = B_2 = 0.4$. $D_1 = D_2 = 0.8$ (as in KMV; all the debt is short term)
Here the dynamics of the firm is a geometric Brownian motion, \( \sigma = 0.25, r = 0.04 \). Leverage = 0.8. T = 15 days. \( B_1 = B_2 = 0.4 \). \( D_1 = D_2 = 0.8 \) (as in KMV; all the debt is short term).
Example 2

The dynamics of the firm is a geometric Brownian motion, \( \sigma = 0.25, \ r = 0.04 \). Leverage = 0.8. \( T = 3 \) months.

\[ D_1 = \ldots = D_8 = 0.1. \ B_1 = \ldots = B_8 = 0.4 \] (short term + 1/3 of long term debt).
The dynamics of the firm is a geometric Brownian motion, $\sigma = 0.25$, $r = 0.04$. Leverage = 0.8. $T = 3$ months.

$D_1 = \ldots = D_8 = 0.1$. $B_1 = \ldots = B_8 = 0.4$ (short term + 1/3 of long term debt).
Conclusion

- New mathematical concept and model for economic default.
- Quantitative model consistent with empirical observation.
- Analytical formula for distribution of $\tau$ and $\tau_e$.
- Arcsine law of the distribution of $\tau - \tau_e$.
- Credit spread is a function of the dynamics of the value of the firm and of the debt structure of the firm,
Thank you for your attention.