A Multiname First Passage Model for Credit Risk

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The Black-Cox Model

- Firm value a geometric Brownian motion
  \[ S_t = S_0 \exp(\mu t + \sigma W_t) \]

- Default threshold deterministic
  \[ B_t = B_0 \exp(\lambda t) \]

- Default is first passage time of \( S_t \) to \( B_t \)
  \[ \tau = \inf \{ t \geq 0 : S_t \leq B_t \} \]
A Closer Look at Black-Cox

- Firms default at FPT of “credit quality” to zero

  \[ X_t^i = \log \left( \frac{S_t^i}{B_t^i} \right) = x_i + \mu_i t + \sigma_i W_t^i \]

- \( W^i \) correlated BM

- \( \mu_i, \sigma_i \) represent trend and volatility in credit quality

- Systematic risk - correlated “noise” about trend
Our Framework

- Model dynamics of credit quality as
  \[ dX^i_t = \mu_i(M_t) \, dt + \sigma_i(V_t) \, dW^i_t \]

- \( M_t, V_t \) correlated processes (unobserved)

- \( \mu_i, \sigma_i \) deterministic functions

- \( W^i \) a BM independent of everything

- Default time \( \tau_i \) is FPT of \( X^i \) to zero
Intuition (Heuristic)

\[
X_{t+h}^i - X_t^i \approx N\left(h\mu_i(M_t), h\sigma_i^2(V_t)\right)
\]

- Systematic factors “set the tone” for a day’s operations

- \(X_{t+h}^i - X_t^i\) and \(X_{t+h}^j - X_t^j\) approximately independent

  - Once the tone has been set, obligors operate independently

- Continuous-time analogue of factor models
General Properties

\[ X_t^i = X_0^i + \int_0^t \mu_i (M_s) \, ds + \int_0^t \sigma_i (V_s) \, dW_s^i \]

- In general credit qualities not Markovian
- Credit qualities are continuous
  - \( M_t, V_t \) may have jumps
- Credit qualities are conditionally independent
Default Process

- Define default process

\[ D_N(t) := \frac{1}{N} \sum_{i=1}^{N} I(\tau_i \leq t) \]

- When it exists, call

\[ D(t) := \lim_{N \to \infty} D_N(t) \]

the asymptotic proportion of defaults
Homogeneous Portfolios

▶ All obligors influenced by systematic factors in same way

\[ dX^i_t = \mu(M_t) \, dt + \sigma(V_t) \, dW^i_t \]

▶ Asymptotic proportion of defaults is

\[ D(t) = P(\tau_i \leq t | \mathcal{H}_t) \]

▶ \( \mathcal{H}_t \) the filtration generated by \( \{ M_s, V_s : 0 \leq s \leq t \} \)

▶ \( D(t) \) is conditional default probability of an arbitrary obligor
A Linear Model

\[ X_t^i = x_0 + Mt + \sqrt{V}W_t^i \]

- \( M, V \) random variables; \( x_0 \) constant

- \( x_0 > 0 \) a constant

- Closed-form for default rate

  \[ D(t) = h(M, V, x_0, t) \]
## Calibration Results

<table>
<thead>
<tr>
<th></th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
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</thead>
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<td>30-100%</td>
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<tr>
<td>CDX</td>
<td>35</td>
<td>34.8</td>
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</tbody>
</table>

- 8 model parameters
- CDS spreads *not* included in calibration
Interesting Observations

- \[ X_t^i = x_0 + Mt + \sqrt{V}W_t^i \]

- Correlation between \( M \) and \( V \) exceeds 80% in both cases
  - 2006 and 2008

- Large portfolio losses (senior tranches impaired) characterized by
  - \( M << 0 \) and \( V \approx 0 \)

- “Low-volatility” market crashes
Interpreting “Low-Volatility” Crashes

- Condition upon \((M, V) = (m, v)\)

\[
X_t^i = x_0 + mt + \sqrt{v}W_t^i
\]

- Now send \(v \to 0\)

\[
X_t^i \approx x_0 + mt
\]

- If \(m < 0\) default with near certainty at \(t^* = -\frac{x_0}{m}\)

  - \(h(m, v, x_0, \cdot)\) converges to degenerate c.d.f. as \(v \to 0\)
$h(-0.375, 0.003, 1.8, \cdot)$
Interpreting the Systematic Factors

\[ X_t^i = x_0 + Mt + \sqrt{V}W_t^i \]

\[ X_t^i \sim Mt \text{ as } t \to \infty \]

- \( M \) is the “dominant long-term” force

- \( V \) modulates influence of idiosyncratic component
  - Downgraded during “bad times”
  - Stochastic “correlation” factor?
Calibrated Densities - $M$

<table>
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<tr>
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<td>403</td>
<td>204</td>
<td>115</td>
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Calibrated Densities - $\log(V)$

- Idiosyncratic risk has been “priced out”
Adding Time Dynamics

\[ dX^i_t = M_t dt + \sqrt{V_t} dW^i_t \quad X^i_0 = x_0 \]

- \( M_t, V_t \) processes with integrable sample paths
- \( x_0 > 0 \) constant

\[ X^i_{t+h} - X^i_t \approx^d N (hM_t, hV_t) \]
Conditional Default Probabilities

- Condition upon realized paths of \((M, V)\), say \((m_t, v_t)\).

\[
X^i_t = x_0 + \int_0^t m_s ds + \int_0^t \sqrt{v_s} dW^i_s
\]

\[
\leq x_0 + \int_0^t m_s ds + W^i \left( \int_0^t v_s ds \right)
\]

\[
= a_t + W^i(b_t)
\]

- Default at first passage of TCBM to non-linear barrier

- \(D(t)\) solves Volterra equation (first kind)
Example

Model $M_t$, $V_t$ as stationary mean-reverting diffusions

\[ dM_t = \theta (\mu - M_t) \, dt + \nu (M_t) \, dZ^1_t \]

\[ dV_t = \alpha (\beta - V_t) \, dt + \xi (V_t) \, dZ^2_t \]

$Z^1, Z^2$ correlated Brownian motion

\[ \nu(\cdot) \text{ chosen so that } M_t \text{ is Laplace} \]

\[ \xi(\cdot) \text{ chosen so that } V_t \text{ is log-Laplace} \]
## Calibration Results - Diffusion Model (2008)

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- 10 model parameters
- Data obtained from Krekel (2008)
- Implied CDS curve is hump-shaped
Comments

- $M_t, V_t$ driven by correlated BM
  - Correlation exceeds 98% in both cases (2006 and 2008)
- Large portfolio losses (senior tranches impaired) characterized by prolonged periods where
  - $M_t \ll 0$ and $V_t \approx 0$
- Unlike linear model, economy can recover
  - Observe cyclical behaviour
The Importance of Time Dynamics

Cumulative Proportion of Defaults

M

V

Cumulative Proportion of Defaults
\[ X_t^i = x_0 + \int_0^t M_s \, ds + \int_0^t \sqrt{V_s} \, dW_s^i \]
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