Dual Pricing of Swing Options with Bang-Bang Control

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Swing Options

- A kind of American-type derivative
- Traded in energy markets, such as gas and electricity
- Characteristics
  - Multiple rights
  - The availability of volume change
    - Option buyer and seller agree to trade energy in the future

- Difficult to price swing options analytically
  - Constraints for changing volume

  Pricing swing options numerically
Previous Works

• Numerical methods for pricing American derivatives
  – lower bounds for the price
    • Least-squares Monte Carlo method
      (Longstaff and Schwartz (01))
    • Extension of LSM to swing options
      (Dörr (03), Barrera-Esteve et al. (06))
  – upper bounds for the price
    • Dual approach
      (Rogers (02), Haugh and Kogan (04), Bender (08), etc.)

• Applying the dual approach in the previous works to swing options is difficult
  – Swing options have flexibility relating to volume
My Contribution

• Extend a dual approach for pricing swing options with bang-bang control
  – Show monotonicity of the optimal exercise strategy
  – Introduce a second-order difference of the price
  – Decompose the pricing problem into single optimal stopping problems
  – Obtain an upper bound
Setup: Swing Options

- Underlying asset price process: $\{X_t\}$
- Possible exercise dates: $t_i$  $(i = 0, 1, \ldots, T)$
- Number of rights: $L$  $(\leq T + 1)$
- All rights must be exercised by $t_T$
- Bang-bang control
  - When a holder exercises a right, he changes traded volume from $u_i$ to 
    \[ u_i + v_{\max} (v_{\min} \leq 0 \leq v_{\max}) \]
    \[ u_i + v_{\min} \]
  - Numbers of choosing $u_i + v_{\max}$ and $u_i + v_{\min}$ are not less than $L_b$ and $L_s$  (buying)  (selling)
Setup: Swing Options (cont’d)

• Constraints can be replaced by rights \((L_b, L_d, L_s)\)
  
  \(L_b\) : Number of obligations to buy  
  \(L_d\) : Number of straddles  
  \(L_s\) : Number of obligations to sell

• Transition tree for the number of rights

\[ (L_b, L_d, L_s) = (1, 1, 2) \]
Formulation of Pricing

- Payoff: 
  \[ Z^b(i) = \nu_{\text{max}}(X_{t_i} - K) \quad (K: \text{strike price}) \]
  \[ Z^s(i) = \nu_{\text{min}}(X_{t_i} - K) \]

- Price of a swing option with \((L_b, L_d, L_s)\) at \(t_i\)
  \[ V(L_b, L_d, L_s, i) = \max [E_i[V(L_b, L_d, L_s, i + 1)], \]
  \[ Z^b(i) + E_i[V(L_b - 1, L_d, L_s, i + 1)], \]
  \[ Z^s(i) + E_i[V(L_b, L_d, L_s - 1, i + 1)] \]

- Optimal strategy on \((L_b, L_d, L_s)\)
  \[ \xi(L_b, L_d, L_s, i) = \begin{cases} 
  \text{"Non-exercise"} & (V(L_b, L_d, L_s, i) = E_i[V(L_b, L_d, L_s, i + 1)]) \\
  \text{"buy"} & (V(L_b, L_d, L_s, i) = Z^b(i) + E_i[V(L_b - 1, L_d, L_s, i + 1)]) \\
  \text{"sell"} & (V(L_b, L_d, L_s, i) = Z^s(i) + E_i[V(L_b, L_d, L_s - 1, i + 1)]) 
\end{cases} \]
Monotonicity for Swing Options

- Optimal strategies between different rights hold monotonicity
- Example on a transition tree

If $\xi(L_b, L_d, L_s - 1, i) = \text{“buy”}$, then $\xi(L_b, L_d, L_s, i) = \text{“buy”}$

If $\xi(L_b, L_d, L_s, i) = \text{“buy”}$, then $\xi(L_b + 1, L_d, L_s - 1, i) = \text{“buy”}$
Dual Approach

- Proposed by Rogers (02) and Haugh and Kogan (04)
- American options price $V(0)$ satisfies
  \[ V(0) = \sup_{0 \leq \tau \leq T} \mathbb{E}[Z(\tau)] \leq \mathbb{E}[\max_{i=0,\ldots,T} (Z(i) - M(i))] \]
  \[(Z(i) : \text{payoff})\]

  for any martingale $M(i)$ ($M(0) = 0$)

- Equality holds for the martingale part $M^*(i)$ of the Doob decomposition of $V(i)$
- An upper bound for the true price can be calculated
Difficulty in Extension for Swing Option

• Multiple rights
  – Natural approach: decomposition into options with a single decision
  – For multiple American options, some studies evaluate a difference for the number of rights

\[ \Delta V(L, 0) = V(L, 0) - V(L - 1, 0) \quad L : \text{number of rights} \]

• Possibility of choice to buy or sell
  – \( \Delta V(L, 0) \) does not reflect the choice \( \rightarrow \) unnatural

How do we decompose?
Introducing Second-Order Difference

• Second-order difference for the number of rights

\[ \Delta \Delta V(L_b, L_d, L_s, i) \equiv V(L_b, L_d, L_s, i) - V(L_b - 1, L_d, L_s, i) - V(L_b, L_d, L_s - 1, i) + V(L_b - 1, L_d, L_s - 1, i) \]

• Price of the swing option can be decomposed into \( \Delta \Delta V(L_b, L_d, L_s, i) \)

Example:

\[
\begin{align*}
V(1, 1, 2, i) &= \sum_{l \in \mathcal{L}(1,1,2)} \Delta \Delta V(l_b, l_d, l_s, i) \\
L(1,1,2) : & \text{ Node set} \\
l : & \text{ Abbreviation of } (l_b, l_d, l_s)
\end{align*}
\]
Introducing Second-Order Difference

- **Second-order difference** for the number of rights
  \[ \Delta \Delta V(L_b, L_d, L_s, i) \equiv V(L_b, L_d, L_s, i) - V(L_b - 1, L_d, L_s, i) - V(L_b, L_d, L_s - 1, i) + V(L_b - 1, L_d, L_s - 1, i) \]

- **Price of the swing option** can be decomposed into \( \Delta \Delta V(L_b, L_d, L_s, i) \)

Example:

Example:

\[ V(1, 1, 2, i) = \sum_{l \in \mathcal{L}(1, 1, 2)} \Delta \Delta V(l_b, l_d, l_s, i) \]

\( \mathcal{L}(1, 1, 2) \) : Node set

\( l \) : Abbreviation of \((l_b, l_d, l_s)\)
Main Result

• Consider optimal stopping problems that correspond to second-order differences \( \Delta \Delta V(l_b, l_d, l_s, i) \)

**Theorem:**

If an exercise strategy \( \xi \) is monotone and a good estimator of the optimal strategy, then it holds that

\[
V(L_b, L_d, L_s, 0) = \sum_{l \in \mathcal{L}(L_b, L_d, L_s)} \Delta \Delta V(l_b, l_d, l_s, 0) \\
\leq \sum_{l \in \mathcal{L}(L_b, L_d, L_s)} \sup_{0 \leq \tau \leq T} \mathbb{E}[Z_l^\xi(\tau)]
\]

\( Z_l^\xi(i) \) : adjusted payoff (discuss later)

• Equality holds for the optimal exercise strategy \( \xi^* \)
Main Result (cont’d)

Theorem:

For any martingale \( M_l(i) \), it holds that

\[
V(L_b, L_d, L_s, 0) = \sum_{l \in \mathcal{L}(L_b, L_d, L_s)} \sup_{0 \leq \tau \leq T} \mathbb{E}[Z_l^\xi(\tau)] \\
\leq \sum_{l \in \mathcal{L}(L_b, L_d, L_s)} \mathbb{E} \left[ \max_{i=0,\ldots,T} [Z_l^\xi(i) - M_l(i)] \right]
\]

- Equality holds for the martingale part \( M_l^*(i) \) of the Doob decomposition of \( \Delta \Delta V(\hat{l}_b^\xi(i), \hat{l}_d^\xi(i), \hat{l}_s^\xi(i), i) \)

\( \hat{l}_b^\xi(i), \hat{l}_d^\xi(i), \hat{l}_s^\xi(i) \): number of residual rights determined by \( \xi \)
Concept of $Z_i^\xi(i)$

- For example, consider $Z$ at $t_0$
- Depend on the number of exercised terms in the second-order difference by $\xi$
  
  **Case 1.** not more than one term
  
  $Z_i^\xi(0) = \max \text{ [payoff from buying, payoff from selling]}

  **Case 2.** two terms
  
  $Z_i^\xi(0) = 0$

  **Case 3.** not less than three terms
  
  $Z_i^\xi(0) = -\infty$

- : strategy $\xi$
  
- : available strategy
Concept of $Z_l^\xi(i)$

- For example, consider $Z$ at $t_0$
- Depend on the number of exercised terms in the second-order difference by $\xi$
  
  Case 1. not more than one term
  \[ Z_l^\xi(0) = \max \text{[payoff from buying, payoff from selling]} \]

  Case 2. two terms
  \[ Z_l^\xi(0) = 0 \]

  Case 3. not less than three terms
  \[ Z_l^\xi(0) = -\infty \]
Concept of $Z^\xi_i(i)$

- For example, consider $Z$ at $t_0$
- Depend on the number of exercised terms in the second-order difference by $\xi$
  - Case 1. not more than one term
    \[ Z^\xi_i(0) = \max \text{[payoff from buying, payoff from selling]} \]
  - Case 2. two terms
    \[ Z^\xi_i(0) = 0 \]
  - Case 3. not less than three terms
    \[ Z^\xi_i(0) = -\infty \]
Numerical Example

• Asset price process: mean-reverting process

$$dX_t = -3 \cdot (X_t - 40)dt + 0.5dW_t, \quad X_{t_0} = 40$$

• \[\begin{cases} \text{Strike price} & K = 40 \\ \text{Maturity} & T = 20 \text{ and } 100 \\ (L_b, L_d, L_s) & : (2, 2, 2), \ (6, 6, 6) \text{ and } (10, 10, 10) \end{cases}\]

Prop.

For a mean-reverting process, the optimal exercise boundary is determined
Numerical Example: Algorithm

• Based on Andersen and Broadie (04) and Bender (08)

Step 1. The least-squares Monte Carlo regression

Step 2. Estimating optimal exercise boundary
   – using coefficients obtained from Step 1

Step 3. Estimating martingales in the theorem
   – from estimated exercise boundary
Numerical Result: Exercise Boundary

![Graph showing exercise boundary with X-axis labeled and time points marked with labels like \((1, 2, 0)\), \((1, 1, 0)\), \((0, 1, 0)\), \((1, 2, 1)\), \((0, 2, 1)\), \((0, 1, 1)\).]
### Numerical Result: Price

<table>
<thead>
<tr>
<th>Rights</th>
<th>$T = 20$</th>
<th>$T = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lower</td>
<td>upper</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>0.8985</td>
<td>0.9007</td>
</tr>
<tr>
<td></td>
<td>(0.0011)</td>
<td>(0.0006)</td>
</tr>
<tr>
<td>(6, 6, 6)</td>
<td>2.0638</td>
<td>2.0692</td>
</tr>
<tr>
<td></td>
<td>(0.0024)</td>
<td>(0.0015)</td>
</tr>
<tr>
<td>(10, 10, 10)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

(Standard errors are in parentheses)

- Differences between upper and lower bounds are less than 1% of the price in all cases
Summary and Future Works

• For the swing option with bang-bang control,
  – the optimal strategy is **monotone**
  – the sum of optimal stopping problems corresponding to **second-order differences** gives an upper bound of the price

• Future works: extension for more complicated options
  – For constant daily and annual constraints, I will be able to extend in a similar way
  – For more general constraints, the dual problem for volume will give an upper bound