Monsoon options: early-exercise Asian tail using fast & accurate hybrid numerical techniques

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Bachelier Finance Congress 2010
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Introduction to the contract

We introduce and value a class of option that is a surprising omission from the current literature, which we christen the ‘Monsoon’ option.

- The number of exercise opportunities, $I$. Unconventionally, the corresponding time to each opportunity does not necessarily equate to the termination of the contract, rather it is simply when a decision may be taken.

- The averaging period for Asian-style options, $\bar{T}$. Best expressed as a fraction of the option’s life, $\bar{T}/T$.

<table>
<thead>
<tr>
<th>Contract</th>
<th>$\bar{T}/T$</th>
<th>$I$</th>
<th>notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monsoon</td>
<td>[0, 1]</td>
<td>$[1, \infty)$</td>
<td>trivial when $I \neq 1$</td>
</tr>
<tr>
<td>- Vanilla Asian</td>
<td>1</td>
<td>$[1, \infty)$</td>
<td></td>
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<tr>
<td>- Asian tail or forward-starting Asian</td>
<td>[0, 1]</td>
<td>1</td>
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<tr>
<td>- European</td>
<td>0</td>
<td>1</td>
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<td>- Bermudan</td>
<td>0</td>
<td>$[1, \infty)$</td>
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<tr>
<td>- American</td>
<td>0</td>
<td>$\infty$</td>
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(Law 2009, Bilger 2004)
Contract space

![Diagram](image)

**Figure:** Schematic of the contract space encompassed by a Monsoon option.
Motivation and modelling scenario

The Monsoon contract encompasses several notable features:

- **Asian features**
  - less susceptible to market manipulation
  - reduce contract volatility - generally cheaper than European

- **Early-exercise**
  - desirable but expensive

Leads us initially to commodities:

- consumed on a continual basis - Asian
- choice of delivery dates - early-exercise

Also note:

- Not to be confused with Hawaiian options (Jørgensen, et al. 1999) - conflicting features
- Bars the use of perpetuals (Chung & Shackleton 2007)
Commodity futures model

Basic model with spot commodity $S$ & convenience yield $y$ (Schwartz 1997)

$$\frac{dS}{S} = (r - y) \, dt + \sigma_S \, dW^Q_S$$

$$dy = (\hat{y} - \kappa_y \, y) \, dt + \sigma_y \, dW^Q_y$$

Leads to futures contract $F(t; T_F)$ maturing at $T_F$ as the underlying (Hilliard & Reis 1998)

$$\frac{dF}{F} = \sigma(t) \, dW^Q_F$$

Time-dependent volatility

$$\sigma(t; T_F) = \sqrt{\sigma_S^2 + \sigma_y^2 B^2 + 2 \rho_{S,y} \sigma_S \sigma_y B}, \quad B(t; T_F) = \frac{1}{\kappa_y} \left( e^{-\kappa_y(T_F-t)} - 1 \right)$$

- We ignore stochastic interest rate.
- Developed further with no-arbitrage model (Trolle & Schwartz 2009).
Monsoon options: early-exercise Asian tail using fast & accurate hybrid numerical techniques
Numerical approach: European example

Four pricing problems, all satisfying Monsoon option criteria:

1. Geometric: Asian tail ... QUAD & Analytic
2. Geometric: Monsoon ... early-exercise QUAD & Analytic
3. Arithmetic: Asian tail ... QUAD & finite-difference
4. Arithmetic: Monsoon ... early-exercise QUAD & finite-difference
Monsoon options: early-exercise Asian tail using fast & accurate hybrid numerical techniques

In order to introduce the notation and valuation approach, begin with simplest option contract

\[ V^E(F, T; T) = \begin{cases} V_{0}^{CE} = [F(T) - K]^+ \\ V_{0}^{PE} = [K - F(T)]^+ \end{cases} \]

Standard hedging arguments with futures (Black 1976) yield

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0 \]

Simply solved by transformations to the heat equation

\[ V^E(x, t; T) = \frac{e^{-r(T-t)}}{2\sqrt{\pi v}} \int_{-\infty}^{\infty} V_0^E(x') \exp \left[ -\frac{(x - x')^2}{4v} \right] dx' \]

We solve using QUAD (Andricopoulos, et al. 2003), with substitution of limits

\[ \hat{x}(F, t_i, t_{i+1}) = x(F, t_i) + D \left( \int_{t_i}^{t_{i+1}} \sigma^2(t')dt' \right)^{1/2} \]

QUAD: Numerical integration specific to derivatives.
Geometric options: Asian tail

1 Geometric options: Asian tail
Geometric options: Asian tail: pricing method

Geometrics are easiest to value, so good starting point

\[
V^{GT}(\bar{F}, T; t^*, T) = \begin{cases} 
V_0^{CGT} = [\bar{F}^{GT}(t^*, T) - K]^+ \\
V_0^{PGT} = [K - \bar{F}^{GT}(t^*, T)]^+
\end{cases}
\]

Start of averaging period \(t^*\); end of averaging and maturity of the contract \(T\). Note that we use \(\bar{T} = T - t^*\).

The geometric average of the tail \(\bar{F}^{GT}(t^*, T)\) given by

\[
\bar{F}^{GT}(t^*, t) = \exp \left[ \frac{1}{t - t^*} \int_{t^*}^{t} \ln(F(t')) dt' \right] \text{ for } t \geq t^*
\]

Solution method - split problem into two regions

- Tail region \(t \in [t^*, T]\), analytic evaluation of the Asian option
- ‘European’ region \(t \in [t_0, t^*]\), solved using QUAD, with ‘payoff’ given by the tail prices at nodes at \(t^*\)
Geometric options: Asian tail: pricing schematic

Figure: Schematic of Asian tail QUAD. Nodes are highlighted to indicate how they are evaluated.
Geometric options: Asian tail: The tail region

Using a convenient definition (Kemna & Vorst 1990)

\[ G(t) = \int_{t^*}^{t} \ln(F(t')) \, dt', \quad G \in (-\infty, \infty) \]

We arrive at the PDE for the fixed strike Asian option price process

\[ \frac{\partial V}{\partial t} + \ln(F) \frac{\partial V}{\partial G} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0 \text{ for } t^* \leq t \leq T \]

Which prices options at \( t^* \) over values of \( F \leftrightarrow x \), i.e. the put option

\[ V^{PGR}(F, t^*; \bar{T}) = e^{-r\bar{T}} \left\{ K\Phi(-d_2) - Fe^{u-w}\Phi(-d_1) \right\} \]

respectively, where \( \Phi(d) \) is the cumulative normal distribution function.
Geometric options: Asian tail: The European region

Identical to the pure European pricing option, with maturity $t^*$, where the payoff is given by the tail.

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0 \text{ for } t_0 \leq t \leq t^*
\]

With the payoff

\[V_0^{GR}(x) = V^{GR}(F, t^*),\]

And thus we can use QUAD with our modified limits of integration to yield

\[V^{GT}(x, t_0; t^*) = \frac{e^{-r(t^*-t_0)}}{2 \sqrt{\pi} v} \int_{\bar{x}(t_0, t^*)}^{\bar{x}(t_0, t^*)} V_0^{GR}(x') \exp \left[ -\frac{(x - x')^2}{4v} \right] dx'\]
2 Geometric options: Monsoon
Geometric options: Monsoon: The tail region

Early-exercise Asian tail - where ‘exercise’ triggers the commencement of the averaging process

\[ V(\bar{F}, t; \bar{T}, T) = \begin{cases} V^{CGM} & \geq [\bar{F}^{GT}(t, t + \bar{T}) - K]^+ \\ V^{PGM} & \geq [K - \bar{F}^{GT}(t, t + \bar{T})]^+ \end{cases} \]

Given early-exercise, we must locate the free-boundary, so the payoff is given by

\[ V_{0}^{GM^*}(x, t_i) = V_{0}^{GR}(x) \text{ if } i = I, \]

\[ V_{0}^{GM^*}(x, t_i) = \left[ V_{0}^{GR}(x), V^{GM}(x, t_i) \right]^+ \text{ if } i < I. \]

The Monsoon contract price is given by

\[ V^{GM}(x, t_i; t_{i+1}) = e^{-r(t_{i+1} - t_i)} \int_{\bar{x}(t_i, t_{i+1})}^{\hat{x}(t_i, t_{i+1})} V_{0}^{GM^*}(x', t_{i+1}) \exp \left[ -\frac{(x - x')^2}{4\nu} \right] dx' \]

- We can curtail local integration to save computing time.
- This describes Bermudan implementation. Richardson extrapolation for Americans. (Andricopoulos et al. 2003)
Geometric options: Monsoon schematic

- analytic Asian evaluation
- free boundary region
- numerical integration

Figure: Schematic of two-step geometric Monsoon QUAD. Nodes are highlighted to indicate how they are evaluated.
3 Arithmetic options: Asian tail
Arithmetic options: Asian tail I

We begin by stating the continuous arithmetic average

$$\bar{F}^A(t) = \frac{1}{t - t^*} \int_{t^*}^{t} F(t') dt' \text{ for } t \geq t^*$$

Neat method to reduce the dimensionality of the problem (Večer 2001, Večer 2002). The identity $d(tf) = tdF + Fdt$ permits

$$\bar{F}^A(t) = F(t^*) + \int_{F(t^*)}^{F(t)} \left(1 - \frac{t'}{t - t^*}\right) dF(t')$$

So we can express the option in terms of a ‘traded account’ $X$, rather than average $\bar{F}$, so $V(F, X, t) = V(F, \bar{F}, t)$, where the account follows

$$X(t) = X(t^*) + \int_{F(t^*)}^{F(t)} q(t') dF(t')$$

given a holding strategy $q(t)$.
Arithmetic options: Asian tail II

We obtain the PDE for the process on the traded account

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial F^2} (\sigma F)^2 + \frac{1}{2} \frac{\partial^2 V}{\partial X^2} (q \sigma F)^2 + \frac{\partial^2 V}{\partial F \partial X} q(\sigma F)^2 - rV = 0
\]

Substitutions \( z(X, F) = \frac{X}{F} \) and \( U(z, t) = \frac{V}{F} \) turn the problem from 3D to 2D

\[
\frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial z^2} \sigma^2 (z - q)^2 - rU = 0
\]

With the boundary conditions

\[
\frac{\partial U}{\partial t} - rU \to 0 \quad \text{as} \quad z \to +\infty; \quad \frac{\partial U}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty
\]

Solve by

- Use of 2D finite-difference grid to price the Asian (tail) region from \( T \) to \( t^* \).
- Grid can value options (payoffs) at \( t^* \) over range of underlying \( F \) in one go.
- Map from grid to QUAD \( U(z) \to V(x) \). Use polynomial interpolation.
Arithmetic options: Asian tail schematic

Figure: Schematic of arithmetic Asian tail QUAD-finite-difference numerical scheme. Nodes are highlighted to show how they are evaluated.
Arithmetic options: Monsoon
Arithmetic options: Monsoon

To extend the Arithmetic method to pricing early-exercise options

- Identical treatment of ‘European’ region, and general method as geometric Monsoon.
- We solve one finite-difference grid per exercise time.
- Locating the free boundary will then only perform polynomial interpolation on the grid, not recalculating.
Results overview

General overview of results is best illustrated by some figures, but

- **General behaviours**
  - Even with short tails, multiple-exercise Monsoons are considerably cheaper than Bermudan equivalent, and often Europeans.

- **Arithmetic vs. Geometric**
  - Geometric average is always smaller than arithmetic (Levy 1992): arithmetic $\neq$ geometric.
  - Given short tails and/or low volatility, geometric method is a practical substitute for Arithmetic.
  - Calculation times for geometric Monsoon options are a matter of seconds, arithmetic less than a minute on modern systems.

Following figures are for contracts with:

- strike $K = 100$
- maturity $T = 3$ months
- absolute price differences shown
Illustration of contract prices

Figure: Geometric put option prices at initiation relative to European. $K = 100$, $T = 0.25$. QUAD numerics: Simpsons, $D = 12$, $x_5 = 10^{-3}$, $\varepsilon = 10^{-9}$. 
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Difference between arithmetic and geometric averaging

Figure: Absolute price premium of geometric put option prices over arithmetic at initiation. QUAD numerics: Simpsons, $D = 12$, $x_\delta = 10^{-3}$, $\varepsilon = 10^{-9}$. FD numerics: $t_\delta = 5 \times 10^{-4}$, $z_\delta = 5 \times 10^{-4}$, $\hat{z} = 2$, $\hat{z} = -2$. Third-order polynomial interpolation.
End & contact

Thank you for listening

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References I


References II


