TWO EXTENSIONS TO

FORWARD START OPTIONS VALUATION

João Pedro Vidal Nunes (ISCTE-IUL Business School, Lisbon)

and

Tiago Ramalho Viegas Alcaria (Commerzbank AG, Frankfurt)

6th World Congress of the Bachelier Finance Society
1. Forward start options.

2. Literature review.

3. Purpose.


5. FFT approach.

6. Direct integration approach (main result).


8. Conclusions.
• Two types of European-style FS (call) options:

\[ c_{\text{FWS}}(T,t^*,T,w) = (S_T - wS_t)^+ \]

\[ c_{\text{RFWS}}(T,t^*,T,w) = (S_T/S_t - w)^+ \]

• Kruse and Nögel (2005):
  – under Heston (1993) SV model; but
  – two 2-dim integrations.

• Mercurio and Moreni (2005): solves integration wrt SV.
• Hong (2004) approach:
  
  – single 1-dim Fourier transform inversion;
  
  – requires characteristic function of the forward rate of return;
  
  – “applicable” to any exponential affine Lévy model;
  
  – BUT requires *model dependent* optimization of a dampening factor $(\alpha)$ to ensure square-integrability.
• Haastrecht and Pelsser (2009):
  
  – Hong (2004) approach under
    
    * SV model of Schöbel and Zhu (1999);
    
    * Gaussian TS model of Hull and White (1993); and
    
    * a full correlation structure.
• Alternative pricing methodology:

  – Valid under the general AJD framework of Duffie, Pan and Singleton (2000);

  – Only requires plain-vanilla option to be homogeneous of degree 1 in spot and strike;

  – Does not require any parallel optimization routine;

  – Yields a single (and exact) Fourier inversion $\implies$ no truncation error;

  – Straightforward to implement (e.g. Gaussian quadrature);

AJD FRAMEWORK

As in Duffie et al. (2000):

- Markovian model factors $X \in D \subseteq \mathbb{R}^n$:

$$dX_t = [K_0 (t) + K_x (t) \cdot X_t] dt + \sigma (X_t, t) \cdot dW^Q_t + dZ^Q_t, \quad (1)$$

$$\sigma (X_t, t) \cdot \sigma (X_t, t)' = H_0 (t) + \sum_{k=1}^{n} H_x^{(k)} (t) (X_t)_k, \quad (2)$$

with $K_0 (t) \in \mathbb{R}^n$, $K_x (t)$, $H_0 (t)$, $H_x^{(k)} (t) \in \mathbb{R}^{n \times n}$.

- Jump-arrival intensity: ($l_0 (t) \in \mathbb{R}$, $l_x (t) \in \mathbb{R}^n$)

$$\lambda (X_t, t) = l_0 (t) + l_x (t)' \cdot X_t. \quad (3)$$

- Short-term interest rate: ($\rho_0 (t) \in \mathbb{R}$, $\rho_x (t) \in \mathbb{R}^n$)

$$r (X_t, t) = \rho_0 (t) + \rho_x (t)' \cdot X_t. \quad (4)$$

8
• Underlying asset $S_t = \exp[(X_t)_1]$ pays continuous (but deterministic) dividend-yield $\delta \in \mathbb{R}$.

• Hence, $X_t = (\ln(S_t), Y_t)$, where $Y_t \in D_y \subset \mathbb{R}^{n-1}$.

• **Assumption 1** (homogeneity requirement):

  $$
  (K_x(t))_{i,1} = \left( H_x^{(1)}(t) \right)_{i,j} = (l_x(t))_1 = (\rho_x(t))_1 = 0, \quad (5)
  $$

  for $i, j = 1, \ldots, n$.

• Very general AJD framework!
Therefore, and based on Duffie et al. (2000, Proposition 1):

$$
\psi(u, t, T; X_t) = \mathbb{E}_Q \left\{ \exp \left[ - \int_t^T r(X_s, s) \, ds \right] \exp \left( u' \cdot X_T \right) \bigg| \mathcal{F}_t \right\} = \exp \left[ \alpha(t, T; u) + u_1 \ln(S_t) + \beta_y(t, T; u)' \cdot Y_t \right]
$$

where

- $u_1$ is the first element of vector $u \in \mathbb{C}^n$; and

- $\beta_y \in \mathbb{C}^{n-1}$ and $\alpha \in \mathbb{C}$ satisfy known complex-valued ODEs.
• Proposition 1 (marginal characteristic functions):

\[
\begin{align*}
    f_j (T, \phi; S_t, Y_t) &= \mathbb{E}_{Q_j} \left[ e^{i\phi \ln(S_T)} | \mathcal{F}_t \right] \\
    &= \exp \left[ \lambda_{c,j} (t, T; \phi) + i\phi \ln (S_t) + \lambda_{y,j} (t, T; \phi)' \cdot Y_t \right],
\end{align*}
\]

(7)

for \( \phi \in \mathbb{C}, \ j = 1, 2, \)

- where \( \lambda_{c,j} (t, T; \phi) \) and \( \lambda_{y,j} (t, T; \phi) \) are simple functions of \( \delta, \beta_y \in \mathbb{C}^{n-1} \) and \( \alpha \in \mathbb{C} \);

- and

<table>
<thead>
<tr>
<th>EMM</th>
<th>Numeraire</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q^S \equiv Q_1 )</td>
<td>( S_t e^{\delta t} )</td>
</tr>
<tr>
<td>( Q_T \equiv Q_2 )</td>
<td>( P(t, T) )</td>
</tr>
</tbody>
</table>
Plain-vanilla options:

- Duffie et al. (2000, Equation 3.5) would involve 2 Fourier transform inversions;


$$c_t(K, T; S_t, Y_t) = S_t e^{-\delta(T-t)} - \frac{KP(t, T)}{2} - K\Omega(t, K, T; S_t, Y_t),$$

where

$$\Omega(t, K, T; S_t, Y_t) = P(t, T) \frac{1}{\pi} \int_0^\infty \text{Re} \left[ e^{-i\phi \ln(K)} f_2(T, \phi; S_t, Y_t) \right] d\phi.$$
**Proposition 2 (Hong (2004)):**

\[
c_{FWS}(t, t^*, T, \omega) = \omega e^{-\delta(T-t)} S_t e^{-\alpha \ln(\omega)} \frac{\pi}{\int_0^\infty e^{-iu \ln(\omega)} g_1(t^*, T, u - i(\alpha - 1); S_t, Y_t) du},
\]

where \(\alpha \in \mathbb{R}_+\), and

\[
g_j(t^*, T, \phi_z; S_t, Y_t) := \mathbb{E}_{Q_j}[e^{i\phi_z z(t^*, T)} \mid \mathcal{F}_t]
\]

is the characteristic function of the forward rate of return

\[
z(t^*, T) := \ln \left( \frac{S_T}{S_{t^*}} \right),
\]

for \(j = 1, 2\) and \(\phi_z \in \mathbb{C}\).
**Proposition 3:** \( g_j(t^*, T, \phi_z; S_t, Y_t) \) can be obtained from the (marginal) characteristic function of the additional state variables \( Y \) (and independently of \( S_t \!):\)

\[
\begin{align*}
h_j(T, \phi_y; Y_t) &= \mathbb{E}_{Q_j} \left( e^{i\phi'_y \cdot Y_T | \mathcal{F}_t} \right) \\
&= \exp \left[ l_{c,j} (t, T; \phi_y) + l_{y,j} (t, T; \phi_y)' \cdot Y_t \right], \tag{12}
\end{align*}
\]

- for \( j = 1, 2 \), where \( \phi_y \in \mathbb{C}^{n-1} \), and

- \( l_{c,j} (t, T; \phi_y) \) and \( l_{y,j} (t, T; \phi_y) \) are simple functions of \( \delta, \beta_y \in \mathbb{C}^{n-1} \) and \( \alpha \in \mathbb{C} \).
DIRECT INTEGRATION APPROACH

• Proposition 4:

\[ c_{FWS} (t, t^*, T, \omega) \]

\[ = S_t e^{\delta_t E_{QS}} \left[ \frac{c_{FWS} (t^*, t^*, T, \omega)}{S_{t^*} e^{\delta_{t^*}}} \right| F_t \]

\[ = S_t e^{-\delta (t^* - t)} E_{QS} \left[ \frac{c_{t^*} (\omega S_{t^*}, T; S_{t^*}, Y_{t^*})}{S_{t^*}} \right| F_t \]

\[ = S_t e^{-\delta (T - t)} - \omega S_t e^{-\delta (t^* - t)} \left\{ \frac{1}{2} E_{QS} [ P (t^*, T) | F_t ] \right. \]

\[ + E_{QS} [ \Omega (t^*, \omega, S_{t^*}, T; S_{t^*}, Y_{t^*}) | F_t ] \right\} \]

\[ = S_t e^{-\delta (T - t)} - \omega S_t e^{-\delta (t^* - t)} \left\{ \frac{1}{2} E_{QS} [ P (t^*, T) | F_t ] \right. \]

\[ + E_{QS} [ \Omega (t^*, \omega, T; 1, Y_{t^*}) | F_t ] \right\}. \]  

(13)
• Proposition 5:

\[
\mathbb{E}_{QS}[\Omega(t^*, \omega, T; 1, Y_{t^*})|\mathcal{F}_t]
\]

\[
= \mathbb{E}_{QS}\left\{ \frac{P(t^*, T)}{\pi} \int_0^\infty \text{Re}\left[ \frac{e^{-i\phi \ln(\omega)} f_2(T, \phi; 1, Y_{t^*})}{\phi^2 + i\phi} \right] d\phi \bigg| \mathcal{F}_t \right\}
\]

\[
= \mathbb{E}_{QS}\left\{ \frac{P(t^*, T)}{\pi} \int_0^\infty \text{Re}\left[ \frac{e^{-i\phi \ln(\omega)}}{(\phi^2 + i\phi) P(t^*, T)} \right] \exp\left( \alpha(t^*, T; (i\phi, 0)) + \beta_y(t^*, T; (i\phi, 0))' \cdot Y_{t^*} \right) d\phi \bigg| \mathcal{F}_t \right\}
\]

\[
= \frac{1}{\pi} \int_0^\infty \text{Re}\left\{ \frac{\exp[\alpha(t^*, T; (i\phi, 0)) - i\phi \ln(\omega)]}{\phi^2 + i\phi} \right\} d\phi.
\]

\[
\mathbb{E}_{QS}\left[ \exp \left( \beta_y(t^*, T; (i\phi, 0))' \cdot Y_{t^*} \right) \bigg| \mathcal{F}_t \right]\}
\]
DIRECT INTEGRATION APPROACH

• Explicit and single 1-dim integral pricing solution (even for $n > 1$);

• Modulo to the specification of $\beta_y(t, T; u) \in \mathbb{C}^{n-1}$ and $\alpha(t, T; u) \in \mathbb{C}$;

• Quadratic term on the denominator $\implies$ fast rate of decay;

• Closed-form solutions for functions $\beta_y(t, T; u) \in \mathbb{C}^{n-1}$ and $\alpha(t, T; u) \in \mathbb{C}$ under the Bakshi, Cao and Chen (1997) model:
  
  – Stochastic volatility; Stochastic interest rates; Jumps in the asset returns;

Numerical Results


- 4 ≠ parameter settings:
  - Bakshi et al. (1997, Table III)—S&P 500 call option prices;
  - Broadie and Kaya (2006, Table 1)—S&P 500 futures option prices;
  - Broadie and Kaya (2006, Table 2)—equity option market;
  - Andersen (2007, Table 1)—long-dated currency options.
NUMERICAL RESULTS

- Proxy for the exact FS option price:
  - Quadratic exponential (and martingale-corrected) Monte Carlo scheme of Andersen (2007);
  - 32 steps per year and $10^7$ paths.

- Proposed direct integration approach:
  - Gauss-Laguerre with 100 weights and abscissas;
    * $[0, \infty) \rightarrow [0, 1]$ following Kahl and Jackel (2006, Equation 41);
    * Relative tolerance of $10^{-12}$. 

19
**NUMERICAL RESULTS**

- Hong (2004) approach:
  - FFT method:
    * Log-strike grid with 16,384 prices and constant spacing of size 0.01.
  - Gauss-Lobatto quadrature is also tested.
  - Extension of the COS approximation of Fang and Oosterlee (2008):
    * Pdf of $z(t^*, T)$ is replaced by its Fourier-cosine series expansion with $10^4$ terms;
    * Same integration range as in Fang and Oosterlee (2008).
Table 1: ATM FS options with $T - t = 2$ years, $t^* - t = 1$ year, $\delta = 0\%$ and $S_t = $100

<table>
<thead>
<tr>
<th>Model setup</th>
<th>Monte Carlo QEM scheme</th>
<th>Propositions 4 and 5</th>
<th>Hong (2004)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho$</td>
<td>$r$</td>
<td>price</td>
</tr>
<tr>
<td>$\kappa_v = 1.15$</td>
<td>-0.64</td>
<td>0.04</td>
<td>8.517</td>
</tr>
<tr>
<td>$\theta_v = 0.04$</td>
<td>-0.9</td>
<td>0.04</td>
<td>8.452</td>
</tr>
<tr>
<td>$\sigma_v = 0.39$</td>
<td>0</td>
<td>0.04</td>
<td>8.617</td>
</tr>
<tr>
<td>$\nu_t = 0.04/1.15$</td>
<td>-0.64</td>
<td>0.10</td>
<td>12.572</td>
</tr>
<tr>
<td>$\kappa_v = 6.21$</td>
<td>-0.7</td>
<td>0.03</td>
<td>6.954</td>
</tr>
<tr>
<td>$\theta_v = 0.11799$</td>
<td>-0.9</td>
<td>0.03</td>
<td>6.940</td>
</tr>
<tr>
<td>$\sigma_v = 0.61$</td>
<td>0</td>
<td>0.03</td>
<td>6.901</td>
</tr>
<tr>
<td>$\nu_t = 0.010201$</td>
<td>-0.7</td>
<td>0.10</td>
<td>11.562</td>
</tr>
<tr>
<td>$\kappa_v = 2$</td>
<td>-0.3</td>
<td>0.05</td>
<td>12.558</td>
</tr>
<tr>
<td>$\theta_v = 0.18$</td>
<td>-0.9</td>
<td>0.05</td>
<td>11.996</td>
</tr>
<tr>
<td>$\sigma_v = 1$</td>
<td>0</td>
<td>0.05</td>
<td>12.774</td>
</tr>
<tr>
<td>$\nu_t = 0.09$</td>
<td>-0.3</td>
<td>0.10</td>
<td>15.504</td>
</tr>
<tr>
<td>$\kappa_v = 0.5$</td>
<td>-0.9</td>
<td>0.00</td>
<td>2.645</td>
</tr>
<tr>
<td>$\theta_v = 0.02$</td>
<td>-0.5</td>
<td>0.00</td>
<td>3.268</td>
</tr>
<tr>
<td>$\sigma_v = 1$</td>
<td>0</td>
<td>0.00</td>
<td>3.927</td>
</tr>
<tr>
<td>$\nu_t = 0.04$</td>
<td>-0.9</td>
<td>0.10</td>
<td>11.087</td>
</tr>
<tr>
<td></td>
<td>-0.9</td>
<td>0.03</td>
<td>5.120</td>
</tr>
</tbody>
</table>

Mean Abs. Percentage Error: 0.020 0.019 0.019 0.019

CPU (seconds): 150879.70 0.05 6.22 1.96 12.66 1.75
Table 2: FS options for different strikes, with $T - t = 2$ years, $t^* - t = 1$ year, $\delta = 0\%$ and $S_t = \$100$

<table>
<thead>
<tr>
<th>Model setup</th>
<th>Monte Carlo QEM scheme</th>
<th>Propositions 4 and 5</th>
<th>Hong (2004)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega$</td>
<td>price</td>
<td>%SE</td>
</tr>
<tr>
<td>$\kappa = 1.15$</td>
<td>0.50</td>
<td>52.036</td>
<td>0.008</td>
</tr>
<tr>
<td>$\theta = 0.04$</td>
<td>0.75</td>
<td>28.662</td>
<td>0.007</td>
</tr>
<tr>
<td>$\sigma = 0.39$</td>
<td>1.00</td>
<td>8.516</td>
<td>0.009</td>
</tr>
<tr>
<td>$v_t = 0.04/1.15$</td>
<td>1.25</td>
<td>0.750</td>
<td>0.059</td>
</tr>
<tr>
<td>$(\rho; r) = (-0.64; 4%)$</td>
<td>1.50</td>
<td>0.098</td>
<td>0.120</td>
</tr>
<tr>
<td>$\kappa = 6.21$</td>
<td>0.50</td>
<td>51.571</td>
<td>0.006</td>
</tr>
<tr>
<td>$\theta = 0.11799$</td>
<td>0.75</td>
<td>27.625</td>
<td>0.006</td>
</tr>
<tr>
<td>$\sigma = 0.61$</td>
<td>1.00</td>
<td>6.954</td>
<td>0.005</td>
</tr>
<tr>
<td>$v_t = 0.010201$</td>
<td>1.25</td>
<td>0.127</td>
<td>0.038</td>
</tr>
<tr>
<td>$(\rho; r) = (-0.7; 3.19%)$</td>
<td>1.50</td>
<td>0.0005</td>
<td>0.230</td>
</tr>
<tr>
<td>$\kappa = 2$</td>
<td>0.50</td>
<td>52.808</td>
<td>0.013</td>
</tr>
<tr>
<td>$\theta = 0.18$</td>
<td>0.75</td>
<td>30.636</td>
<td>0.013</td>
</tr>
<tr>
<td>$\sigma = 1$</td>
<td>1.00</td>
<td>12.556</td>
<td>0.016</td>
</tr>
<tr>
<td>$v_t = 0.09$</td>
<td>1.25</td>
<td>3.804</td>
<td>0.032</td>
</tr>
<tr>
<td>$(\rho; r) = (-0.3; 5%)$</td>
<td>1.50</td>
<td>1.415</td>
<td>0.052</td>
</tr>
<tr>
<td>$\kappa = 0.5$</td>
<td>0.50</td>
<td>50.204</td>
<td>0.006</td>
</tr>
<tr>
<td>$\theta = 0.02$</td>
<td>0.75</td>
<td>25.839</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma = 1$</td>
<td>1.00</td>
<td>2.644</td>
<td>0.039</td>
</tr>
<tr>
<td>$v_t = 0.04$</td>
<td>1.25</td>
<td>0.232</td>
<td>0.187</td>
</tr>
<tr>
<td>$(\rho; r) = (-0.9; 0%)$</td>
<td>1.50</td>
<td>0.081</td>
<td>0.198</td>
</tr>
</tbody>
</table>

Mean Abs. Percentage Error 0.179 0.218 0.224 0.218 0.218
MAPE (full truncation Euler MC) 0.109 0.068 0.074 0.069 0.068
CPU (seconds) 77859.93 0.08 7.83 1.95 41.49 1.78
Table 3: FS options for different times to determination and to maturity, with $\delta = 0\%$ and $S_t = $100

<table>
<thead>
<tr>
<th>Model setup</th>
<th>Monte Carlo Propositions 4 and 5</th>
<th>Hong (2004)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QEM scheme</td>
<td>G-Laguerre</td>
</tr>
<tr>
<td>$\kappa_v = 1.15$</td>
<td>t $^*$ $-$ t</td>
<td>$\tau$</td>
</tr>
</tbody>
</table>
|             | 0.0625 | 0.5 | 5.495 | 0.005 | 0.002 | 0.002 | 0.002 | 0.002 
| $\theta_v = 0.04$ | 0.2500 | 0.5 | 3.785 | 0.010 | -0.001 | -0.001 | -0.001 | -0.001 |
| $\sigma_v = 0.39$ | 0.4375 | 0.5 | 1.663 | 0.015 | -0.062 | -0.078 | -0.045 | -0.078 |
| $v_t = 0.04/1.15$ | 0.6250 | 5.0 | 23.276 | 0.004 | -0.001 | -0.001 | -0.001 | -0.001 |
| $\rho = -0.64$ | 2.5000 | 5.0 | 15.797 | 0.008 | -0.067 | -0.067 | -0.067 | -0.067 |
| $r = 4\%$ | 4.3750 | 5.0 | 6.172 | 0.012 | -0.122 | -0.122 | -0.122 | -0.122 |
| $\kappa_v = 6.21$ | 0.0625 | 0.5 | 3.950 | 0.003 | 0.000 | 0.000 | 0.000 | 0.000 |
| $\theta_v = 0.11799$ | 0.2500 | 0.5 | 2.837 | 0.006 | 0.008 | 0.008 | 0.008 | 0.008 |
| $\sigma_v = 0.61$ | 0.4375 | 0.5 | 1.257 | 0.013 | -0.010 | -0.008 | -0.008 | -0.008 |
| $v_t = 0.010201$ | 0.6250 | 5.0 | 18.615 | 0.003 | -0.007 | -0.007 | -0.007 | -0.007 |
| $\rho = -0.7$ | 2.5000 | 5.0 | 12.757 | 0.006 | -0.011 | -0.011 | -0.011 | -0.011 |
| $r = 3.19\%$ | 4.3750 | 5.0 | 5.129 | 0.009 | -0.111 | -0.111 | -0.111 | -0.111 |
| $\kappa_v = 2$ | 0.0625 | 0.5 | 8.004 | 0.007 | -0.009 | -0.009 | -0.009 | -0.009 |
| $\theta_v = 0.18$ | 0.2500 | 0.5 | 5.490 | 0.014 | 0.011 | 0.011 | 0.011 | 0.011 |
| $\sigma_v = 1$ | 0.4375 | 0.5 | 2.412 | 0.022 | 0.023 | 0.017 | 0.024 | 0.016 |
| $v_t = 0.09$ | 0.6250 | 5.0 | 32.145 | 0.007 | -0.007 | -0.007 | -0.007 | -0.007 |
| $\rho = -0.3$ | 2.5000 | 5.0 | 22.627 | 0.015 | -0.076 | -0.076 | -0.076 | -0.076 |
| $r = 5\%$ | 4.3750 | 5.0 | 9.270 | 0.022 | -0.096 | -0.096 | -0.096 | -0.096 |
| $\kappa_v = 0.5$ | 0.0625 | 0.5 | 3.028 | 0.023 | -0.324 | -0.297 | -0.291 | -0.297 |
| $\theta_v = 0.02$ | 0.2500 | 0.5 | 1.796 | 0.040 | -0.457 | -0.144 | -0.251 | -0.144 |
| $\sigma_v = 1$ | 0.4375 | 0.5 | 0.734 | 0.096 | 3.321 | 0.390 | 1.263 | 0.389 |
| $v_t = 0.04$ | 0.6250 | 5.0 | 7.080 | 0.009 | -0.031 | -0.031 | -0.031 | -0.031 |
| $\rho = -0.9$ | 2.5000 | 5.0 | 4.495 | 0.013 | 0.000 | 0.000 | 0.000 | 0.000 |
| $r = 0\%$ | 4.3750 | 5.0 | 1.765 | 0.034 | -0.103 | -0.138 | -0.139 | -0.138 |

Mean Abs. Percent. Error | 0.198 | 0.064 | 0.103 | 0.064 | 0.072 |

CPU (seconds) | 190814.54 | 0.07 | 2.70 | 2.41 | 126.02 | 2.05 |
Speed-accuracy trade-off
• The COS approximation can be biased in a low mean reversion setting.

• The QEM Monte Carlo scheme can be biased for deep out-of-the-money contracts.

• The adaptive Gauss-Lobatto quadrature scheme is the most robust integration method.

• The direct integration method proposed provides a better accuracy-efficiency trade-off than the usual Hong (2004) approach.
References


Hong, G., 2004, Forward Smile and Derivative Pricing, Working paper, UBS.


