Explicit Representation of Cost Efficient Strategies

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Starting point: work on popular US retail investment products. How to explain the demand for complex path-dependent contracts?


Path-dependent contracts are not “efficient” (JoB 1988, “Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market” in RFS 1988).
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**Met with Phil Dybvig at the NFA in Sept. 2008.**

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Some Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff $X_T$ at time $T$. There exists $Q$, such that its price at 0 is
  \[ P_X = E_Q[e^{-rT} X_T] \]
- $P$ ("physical measure") and $Q$ ("risk-neutral measure") are two equivalent probability measures:
  \[ \xi_T = e^{-rT} \left( \frac{dQ}{dP} \right)_T, \quad P_X = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T]. \]
Motivation: Traditional Approach to Portfolio Selection

Investors have a strategy that will give them a final wealth $X_T$. This strategy depends on the financial market and is random.

- They want to maximize the \textbf{expected utility} of their final wealth $X_T$

$$\max_{X_T} (E_P[U(X_T)])$$

$U$: utility (increasing because individuals prefer more to less).

- They want to minimize the \textbf{cost of the strategy}

$$\text{cost at } 0 = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T]$$

Find optimal payoff $X_T$ $\Rightarrow$ Optimal cdf $F$ of $X_T$
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**Find optimal payoff** $X_T \Rightarrow \text{Optimal cdf } F$ of $X_T$
Our Approach

- Given the cdf $F$ that the investor would like for his final wealth

- We give an explicit representation of the payoff $X_T$ such that
  - $X_T \sim F$ in the real world
  - $X_T$ corresponds to the cheapest strategy
Outline of the presentation

- What is cost-efficiency?
- Path-dependent strategies/payoffs are not cost-efficient.
- **Explicit** construction of efficient strategies.
- Investors (with a fixed horizon and law-invariant preferences) should prefer to invest in path-independent payoffs: path-dependent exotic derivatives are usually not optimal!
Efficiency Cost

Dybvig (RFS 1988) explains how to compare two strategies by analyzing their respective efficiency cost.

What is the “efficiency cost”?

It is a criteria for evaluating payoffs independent of the agents’ preferences.
Efficiency Cost

- Given a strategy with payoff $X_T$ at time $T$, and initial price at time 0

$$P_X = E_P [\xi_T X_T]$$

- $F : X_T$’s distribution under the physical measure $P$.

The distributional price is defined as

$$PD(F) = \min_{\{Y_T \mid Y_T \sim F\}} \{E_P [\xi_T Y_T]\}$$

The “loss of efficiency” or “efficiency cost” is equal to:

$$P_X - PD(F)$$
A Simple Illustration

Let’s illustrate what the “efficiency cost” is with a simple example. Consider:

- A market with 2 assets: a bond and a stock $S$.
- A discrete 2-period binomial model for the stock $S$.
- A strategy with payoff $X_T$ at the end of the two periods.
- An expected utility maximizer with utility function $U$. 
A simple illustration for $X_2$, a payoff at $T = 2$

**Real-world** probabilities $p = \frac{1}{2}$

\[
E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2}
\]
$Y_2$, a payoff at $T = 2$ distributed as $X_2$

Real-world probabilities $= p = \frac{1}{2}$

$$E[U(Y_2)] = \frac{U(3) + U(1)}{4} + \frac{U(2)}{2}$$

($X$ and $Y$ have the same distribution under the physical measure and thus the same utility)
$X_2$, a payoff at $T = 2$

**risk neutral**

probabilities $q = \frac{1}{4}$.

\[
\begin{align*}
S_0 &= 16 \\ S_1 &= 8 \\ S_1 &= 32 \\ S_2 &= 64 \\ S_2 &= 16 \\ S_2 &= 16 \\ &\quad \text{S}_2 = 4
\end{align*}
\]

\[
P_{X_2} = \text{Price of } X_2 = \left( \frac{1}{16} \right) + \left( \frac{6}{16} \right)^2 + \left( \frac{9}{16} \right)^3 = \frac{5}{2}
\]
Y₂, a payoff at \( T = 2 \)

Risk neutral probabilities: \( q = \frac{1}{4} \).

\[
P_{Y_2} = \left( \frac{1}{16} \cdot 3 + \frac{6}{16} \cdot 2 + \frac{9}{16} \cdot 1 \right) = \frac{3}{2}
\]

\[
P_{X_2} = \text{Price of } X_2 = \left( \frac{1}{16} + \frac{6}{16} \cdot 2 + \frac{9}{16} \cdot 3 \right) = \frac{5}{2}
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P_D = \text{Cheapest} = \frac{3}{2}
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A simple illustration for $X_2$, a payoff at $T = 2$

Risk neutral probabilities $q = \frac{1}{4}$.

$S_2 = 64$

$S_1 = 32$

$S_0 = 16$

$S_1 = 8$

$S_2 = 4$

$S_2 = 16$

$P_D = \text{Cheapest} = \frac{3}{2}$

$P_{X_2} = \text{Price of } X_2 = \frac{5}{2}$, Efficiency cost $= P_{X_2} - P_D$
A simple illustration for $X_2$, a payoff at $T = 2$

Real-world probabilities $= p = \frac{1}{2}$ and risk neutral probabilities $= q = \frac{1}{4}$.

$$E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2}, \quad P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \frac{5}{2}, \quad \text{Efficiency cost} = P_{X_2} - P_D$$
Cost-Efficiency

- The cost of the payoff $X_T$ is $c(X_T) = E[\xi_T X_T]$.
- The "distributional price" of a cdf $F$ is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{c(Y)\}$$

We want to find the strategy $Y$ that realizes this minimum.

Given a payoff $X_T$ with cdf $F$. We define its inverse $F^{-1}$ as follows:

$$F^{-1}(y) = \min \{x \mid F(x) \geq y\}.$$

**Theorem**

Define

$$X_T^* = F^{-1} (1 - F_\xi (\xi_T))$$

then $X_T^* \sim F$ and $X_T^*$ is a.s. unique such that

$$PD(F) = c(X_T^*)$$
Path-dependent payoffs are inefficient

Corollary

To be cost-efficient, the payoff of the derivative has to be of the following form:

\[ X_T^* = F^{-1} (1 - F_{\xi}(\xi_T)) \]

It becomes a European derivative written on \( S_T \) as soon as the state-price process \( \xi_T \) can be expressed as a function of \( S_T \). Thus path-dependent derivatives are in general not cost-efficient.

Corollary

Consider a derivative with a payoff \( X_T \) which could be written as

\[ X_T = h(\xi_T) \]

Then \( X_T \) is cost efficient if and only if \( h \) is non-increasing.
Black and Scholes Model

Under the physical measure $P$,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Under the risk neutral measure $Q$,

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t^Q$$

$S_t$ has a lognormal distribution.

$$\xi_T = e^{-rT} \left( \frac{dQ}{dP} \right)_T = e^{-rT} a \left( \frac{S_T}{S_0} \right)^{-b}$$

where $a = \exp \left( \frac{1}{2} T b(r + \mu - \sigma^2) - rT \right)$, $b = \frac{\mu - r}{\sigma^2}$. 

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Path-dependent inefficient strategies 19
Black and Scholes Model

Any path-dependent financial derivative is inefficient. Indeed

\[ \xi_T = e^{-rT} \left( \frac{dQ}{dP} \right)_T = e^{-rT} a \left( \frac{S_T}{S_0} \right)^{-b} \]

where \( a = \exp \left( \frac{1}{2} T b \left( r + \mu - \sigma^2 \right) - r T \right) \)

To be cost-efficient, the payoff has to be written as

\[ X^* = F^{-1} \left( 1 - F_\xi \left( a \left( \frac{S_T}{S_0} \right)^{-b} \right) \right) \]

It is a European derivative written on the stock \( S_T \) (and the payoff is increasing with \( S_T \) when \( \mu > r \)).
The Least Efficient Payoff

Theorem

Let $F$ be a cdf such that $F(0) = 0$. Consider the following optimization problem:

$$\max \{ c(Z) \mid Z \sim F \}$$

The strategy $Z_T^\star$ that generates the same distribution as $F$ with the highest cost can be described as follows:

$$Z_T^\star = F^{-1}(F_\xi(\xi_T))$$

Consider a strategy with payoff $X_T$ distributed as $F$. The cost of this strategy satisfies

$$P_D(F) \leq c(X_T) \leq E[\xi_T F^{-1}(F_\xi(\xi_T))] = \int_0^1 F_\xi^{-1}(v) F^{-1}(v) dv$$
Put option in Black and Scholes model

Assume a strike $K$. The payoff of the put is given by

$$L_T = (K - S_T)^+.\]

The payoff that has the lowest cost and is distributed such as the put option is given by

$$Y_T^* = F_L^{-1} (1 - F_\xi (\xi_T)).\]
Put option in Black and Scholes model

Assume a strike $K$. The payoff of the put is given by

$$L_T = (K - S_T)^+.\$$

The cost-efficient payoff that will give the same distribution as a put option is

$$Y^*_T = \left( K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T} \right)^+.$$

This type of power option “dominates” the put option.
Cost-efficient payoff of a put

With $\sigma = 20\%$, $\mu = 9\%$, $r = 5\%$, $S_0 = 100$, $T = 1$ year, $K = 100$. 
Distributional price of the put = 3.14
Price of the put = 5.57
Efficiency loss for the put = 5.57-3.14= 2.43
**Geometric Asian contract in Black and Scholes model**

Assume a strike $K$. The payoff of the Geometric Asian call is given by

$$G_T = \left( e^{\frac{1}{T} \int_0^T \ln(S_t) dt} - K \right)^+$$

which corresponds in the discrete case to

$$\left( \left( \prod_{k=1}^n S_{kT/n} \right)^{1/n} - K \right)^+.$$  

The efficient payoff that is distributed as the payoff $G_T$ is given by

$$G_T^* = d \left( S_T^{1/\sqrt{3}} - \frac{K}{d} \right)^+$$

where

$$d := S_0^{1 - \frac{1}{\sqrt{3}}} e^{\left( \frac{1}{2} - \frac{1}{\sqrt{3}} \right) \left( \mu - \frac{\sigma^2}{2} \right) T}.$$  

This payoff $G_T^*$ is a power call option. If $\sigma = 20\%$, $\mu = 9\%$, $r = 5\%$, $S_0 = 100$. The price of this geometric Asian option is 5.94. The payoff $G_T^*$ costs only 5.77.  

Similar result in the discrete case.
Example: the discrete Geometric option

With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100, T = 1$ year, $K = 100, n = 12$. Price of the geometric Asian option = 5.94. The distributional price is 5.77. The least-efficient payoff $Z_T^*$ costs 9.03.
Utility Independent Criteria

Denote by

- $X_T$ the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions

1. **Agents’ preferences depend only on the probability distribution of terminal wealth**: “law-invariant” preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)

2. **Agents prefer “more to less”**: if $c$ is a non-negative random variable $V(X_T + c) \geq V(X_T)$.

3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).

4. The market is **arbitrage-free** and **complete**.

For any inefficient payoff, there exists another strategy that these agents will prefer.
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Link with First Stochastic Dominance

Theorem

Consider a payoff $X_T$ with cdf $F$,

1. **Taking into account the initial cost of the derivative, the cost-efficient payoff $X_T^*$ of the payoff $X_T$ dominates $X_T$ in the first order stochastic dominance sense:**

   $$X_T - c(X_T)e^{rT} \prec_{fsd} X_T^* - P_D(F)e^{rT}$$

2. **The dominance is strict unless $X_T$ is a non-increasing function of $\xi_T$.**

Thus the result is true for any preferences that respect first stochastic dominance.
Explaining the Demand for Inefficient Payoffs

1. State-dependent needs
   - **Background risk:**
     - Hedging a long position in the market index $S_T$ (background risk) by purchasing a put option $P_T$,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark $\Gamma$ such that:
     \[ P \{ \omega \in \Omega / W_T(\omega) > \Gamma(\omega) \} \geq \alpha. \]
   - **Intermediary consumption.**

2. Other sources of uncertainty: the state-price process is not always a monotonic function of $S_T$ (non-Markovian interest rates for instance)

3. Transaction costs, frictions: Preference for an available inefficient contract rather than a cost-efficient payoff that one needs to replicate.
Conclusion

- A preference free framework for ranking different investment strategies.
- For a given investment strategy, we derive an explicit analytical expression
  1. for the cheapest strategy that has the same payoff distribution.
  2. for the most expensive strategy that has the same payoff distribution.
- There are strong connections between this approach and stochastic dominance rankings.

This may be useful for improving the design of financial products.