Asymptotic Analysis for Optimal Investment with Two Risky Assets and Transaction Costs

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History

- Merton, J. Econ, 1971
  - Optimal consumption and investment problem in continuous time.
- Magill & Constantinides, JET, 1976
  - Proportional transaction costs in Merton’s model.
- Davis & Norman, Mathematics of Operations Research, 1990
  - Rigorous treatment of Magill & Constantinides model
  - Existence and uniqueness of solution to the HJB equation
  - Shape of the optimal policy.
  - Viscosity solution analysis of Magill & Constantinides model
  - Smoothness of the value function.
- Whalley & Wilmott, Mathematical Finance 1997
  - Pricing an option
  - Asymptotic expansion of the value function in powers of $\lambda^{\frac{1}{3}}$. 
History (cont)

- Janeček & Shreve, Finance and Stochastics, 2004
  - Model with one stock
  - Viscosity solution approach to compute the loss in asymptotic expansion of the value function.

- Atkinson & Ingpochai J. of Comp Fin. 2007
  - Multiple assets
  - Loss in asymptotic expansion of the value function
  - Asymptotically correlated assets.
Buyer of a Futures contract receives changes in futures price.

Change in futures price of futures contract of type $i$

$$dF_i(t) = \mu_i dt + \sigma_i dB_i(t),$$

where $<B_1, B_2>_t = \rho t$.

Number of futures contracts of type $i$ held at time $t$

$$Y_i(t) = y_i + L_i(t) - M_i(t).$$

Change in cash held at time $t$

$$dX(t) = \sum_{i=1}^{2} Y_i(t) dF_i(t) - \sum_{i=1}^{2} \lambda (dL_i(t) + dM_i(t)) + X(t)(r - c(t)) dt.$$
\[ S_v = \{(y_1, y_2, x) : x - \lambda|y_1| - \lambda|y_2| > 0\} . \]

The strategy \((c, L_i, M_i)\) is admissible for \((y_1, y_2, x)\), if \((Y_1(t), Y_2(t), X(t)) \in \overline{S_v}\) for all \(t \geq 0\).

\[ U(c) \triangleq \frac{c^p}{p}, \ c \geq 0, \ 0 < p < 1. \]

Max the expected integral of the discounted utility of consumption

\[ v(y_1, y_2, x) = \sup_{\text{Admissible strategies}} E \left[ \int_0^\infty e^{-\beta t} U(X(t)c(t)) dt \right]. \]
Properties of the value function

\[ v \bigg|_{\partial S_v} = 0. \]

\( v \) is homogeneous of degree \( p \):  
\[ \forall \alpha > 0 : \ v(\alpha y_1, \alpha y_2, \alpha x) = \alpha^p v(y_1, y_2, x). \]

Reduction of variables

\[ u(z_1, z_2) \triangleq v(z_1, z_2, 1), \]
\[ v(y_1, y_2, x) = x^p v \left( \frac{y_1}{x}, \frac{y_1}{x}, 1 \right) = x^p u(z_1, z_2), \quad z_i = \frac{y_i}{x}. \]
Merton’s Model results

\[ \lambda = 0 \]

Optimal investment proportion:

\[ \theta_i = \frac{\mu_i \sigma_j - \rho \mu_j \sigma_i}{(1-p)\sigma_i^2 \sigma_j (1-\rho^2)}, \]

\[ A = \frac{\beta - rp}{1-p} - \frac{p}{1-p} \sum_{i=1}^{2} \mu_i \theta_i + \frac{p}{2} \left( \sigma_1^2 \theta_1^2 + \sigma_1^2 \theta_1^2 + 2\rho \sigma_1 \sigma_2 \theta_1 \theta_2 \right). \]

Standing Assumption

\[ A > 0. \]

\[ \lambda = 0 \]

\[ u(z_1, z_2) = \frac{1}{p} A^{p-1}, \]

Optimal consumption proportion \( c(t) = A. \)
HJB equation \((\lambda > 0)\)

\[
\tilde{U}(\tilde{c}) \triangleq \sup_{c>0} \{ U(c) - c\tilde{c} \} = \frac{1 - p\tilde{c}}{p} - \frac{p}{1-p}.
\]

**HJB equation**

\[
\min \left[ \mathcal{D}(u) - \tilde{U}(\rho u - z_1u_1 - z_2u_2) , \mathcal{B}_1(u), \mathcal{S}_1(u), \mathcal{B}_2(u), \mathcal{S}_2(u) \right] = 0.
\]

- **What is** \(\mathcal{D}(u)\)?
  - Second order diffusion operator.
  - Diffusion in radial direction only.

- **What are** \(\mathcal{B}_1(u), \mathcal{S}_1(u), \mathcal{B}_2(u), \mathcal{S}_2(u)\)?
  - First order differential operators.
  - Their characteristics correspond to pure trades.
Division of solvency region

\[ B_1(u) = 0 \]

\[ S_1(u) = 0 \]

\[ B_2(u) = 0 \]

\[ S_2(u) = 0 \]
Main Theorem 1, when \( \rho = 0 \)

**Theorem**

Assume \( 0 < p < 1, \ A > 0, \ \lambda > 0 \) and \( \rho = 0 \). Fix \((z_1, z_2) \in S_u\), then the value function is

\[
u(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2 \lambda^\frac{2}{3} + O(\lambda),
\]

where

\[
\gamma_2 = A^{p-2} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32}} (1 - p) \sigma_i^2 \theta_i^4 (\sigma_1^2 \theta_1^2 + \sigma_2^2 \theta_2^2)^2.
\]
Lemma

Assume $0 < p < 1$, $A > 0$, $\lambda > 0$ and $\rho = 0$. Then there exist $NT^\pm$ regions, such that if inside we define:

$$w^\pm(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2 \lambda^2 \pm M \lambda$$

$$- \sum_{i=1}^{2} \frac{A^{p-1}}{\nu_i} \left( \frac{3}{2} (z_i - \theta_i)^2 \lambda^2 - \frac{1}{\nu_i^2} (z_i - \theta_i)^4 + \frac{3}{2} B (z_i - \theta_i)^2 \lambda^4 \right),$$

and define $w^\pm$ in the rest of the solvency region $S_u$ using characteristics. Then $w^\pm$ is a super/sub-solution of the HJB equation with zero boundary condition.
Heuristics, when $\rho \neq 0$

What goes wrong?

If one assumes that the value function $u$ has a power expansion and is $C^2(S_u)$, then the system of equations obtained by equating the derivatives across the boundary of the NT region is inconsistent.
An auxiliary model, when $\rho > 0$

Consider a new set of futures contracts:

$$\tilde{F}_1(t) = F_1(t), \quad \tilde{F}_2(t) = F_2(t) - \rho \frac{\sigma_2}{\sigma_1} F_1(t).$$

To write them in the original form $\tilde{F}_i(t) = \tilde{\mu}_i t + \tilde{\sigma}_i \tilde{B}_i(t)$, define:

$$\tilde{B}_1(t) = B_1(t), \quad \tilde{B}_2(t) = \frac{1}{\sqrt{1 - \rho^2}} (B_2(t) - \rho B_1(t)).$$

$$\tilde{\sigma}_1 = \sigma_1, \quad \tilde{\sigma}_2 = \sigma_2 \sqrt{1 - \rho^2},$$

$$\tilde{\mu}_1 = \mu_1, \quad \tilde{\mu}_2 = \mu_2 - \rho \frac{\sigma_2}{\sigma_1} \mu_1,$$

$$\tilde{\lambda}_1 = \lambda, \quad \tilde{\lambda}_2 = \left(1 - \rho \frac{\sigma_2}{\sigma_1}\right) \lambda.$$
Lemma

Assume $0 < \rho < 1$, $A > 0$, $\lambda > 0$ and $0 < \rho < \frac{\sigma_1}{\sigma_2}$. Let $\tilde{v}$ be the value function associated with the problem with the new set of futures contracts. Then for any $(y_1, y_2, x) \in S_v$ we have that $v(y_1, y_2, x) \leq \tilde{v}(\tilde{y}_1, \tilde{y}_2, x)$, where $\tilde{y}_2 = y_2$, $\tilde{y}_1 = y_1 + \rho \frac{\sigma_2}{\sigma_1} y_2$.

Similarly define: $\tilde{S}_u = \left\{ (\tilde{z}_1, \tilde{z}_2) : 1 - \tilde{\lambda}_1 |\tilde{z}_1| - \tilde{\lambda}_2 |\tilde{z}_2| > 0 \right\}$,

$\tilde{u}(\tilde{z}_1, \tilde{z}_2) = \tilde{v}(\tilde{z}_1, \tilde{z}_2, 1)$, $(\tilde{z}_1, \tilde{z}_1) \in \tilde{S}_u$.

Corollary

Under the above assumptions, a viscosity super-solution for $\tilde{u}$ is an upper bound for $u$. 
- Auxiliary model not identical to the original model.
- Subsolution.
- Construct on subset of $NT^-$ region and then extend.
- Uses the fact that diffusion is in the radial direction.
Main Theorem 2, when \( 0 < \rho < \frac{\sigma_1}{\sigma_2} \)

**Theorem**

Assume \( 0 < p < 1, \ A > 0, \ \lambda > 0, \ \mu_1 > \mu_2, \ \theta_1, \ \theta_2 > 0 \) and \( 0 < \rho < \frac{\sigma_1}{\sigma_2} \).

Fix \((z_1, z_2) \in S_u\), then the value function is

\[
u(z_1, z_2) = \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^2 + O(\lambda),
\]

where

\[
\tilde{\gamma}_2 = A^{p-2} \sum_{i=1}^{2} \sqrt{\frac{9}{32} \left( \frac{\tilde{\lambda}_i}{\lambda} \right)^2 (1 - p) \tilde{\sigma}_i^2 \tilde{\theta}_i^4 \left( \tilde{\sigma}_1^2 \tilde{\theta}_1^2 + \tilde{\sigma}_2^2 \tilde{\theta}_2^2 \right)^2}.
\]
Thank you!

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